

# Similarity degree of a class of $C^*$ -algebras

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**ABSTRACT.** Suppose that  $\mathcal{M}$  is a countably decomposable type  $II_1$  von Neumann algebra and  $\mathcal{A}$  is a separable, non-nuclear, unital  $C^*$ -algebra. We show that, if  $\mathcal{M}$  has Property  $\Gamma$ , then the similarity degree of  $\mathcal{M}$  is less than or equal to 5. If  $\mathcal{A}$  has Property  $c^*\Gamma$ , then the similarity degree of  $\mathcal{A}$  is equal to 3. In particular, the similarity degree of a  $\mathcal{Z}$ -stable, separable, non-nuclear, unital  $C^*$ -algebra is equal to 3.

## 1. Introduction

Kadison's Similarity Problem for a  $C^*$ -algebra  $\mathcal{A}$  in [15] asks whether every bounded representation  $\rho$  of  $\mathcal{A}$  on a Hilbert space  $H$  is similar to a  $*$ -representation. i.e. whether there exists an invertible operator  $T$  in  $B(H)$ , such that  $T\rho(\cdot)T^{-1}$  is a  $*$ -representation of  $\mathcal{A}$ .

In [3], Christensen proved that every irreducible bounded representation of a  $C^*$ -algebra on a Hilbert space is similar to a  $*$ -representation. He also showed that every non-degenerate bounded representation of a nuclear  $C^*$ -algebra is similar to a  $*$ -representation (also see [2]).

In [9], Haagerup showed that every cyclic (or finitely cyclic) bounded representation of a  $C^*$ -algebra on a Hilbert space is similar to a  $*$ -representation. From this, he concluded that, if  $\mathcal{A}$  is a  $C^*$ -algebra that has no tracial states, then every non-degenerate bounded representation of  $\mathcal{A}$  is similar to a  $*$ -representation. It was also shown in [9] that a non-degenerate bounded representation  $\rho$  of a  $C^*$ -algebra  $\mathcal{A}$  on a Hilbert space  $H$  is similar to a  $*$ -representation if and only if  $\rho$  is completely bounded (also see [10], [27]).

From Haagerup's results, it follows that Kadison's Similarity Problem remains only open for  $C^*$ -algebras with tracial states. Since a type  $II_1$  von Neumann algebra always has tracial states, it is natural to consider Kadison's Similarity Problem for von Neumann algebras of type  $II_1$ . In [5], Christensen showed that Kadison's Similarity Problem for type  $II_1$  factors with Property  $\Gamma$  has an affirmative answer.

In order to study Kadison's Similarity Problem, Pisier in [18] introduced a powerful new concept, similarity degree of a unital  $C^*$ -algebra, as follows. *Suppose  $\mathcal{A}$  is a unital  $C^*$ -algebra. It has finite similarity degree if there is  $\alpha > 0$  such that for some constant  $k$  (depending on  $\alpha$ )*

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we have, for every bounded unital representation  $\phi$  of  $\mathcal{A}$  on a Hilbert space  $H$ ,

$$\|\phi\|_{cb} \leq k\|\phi\|^\alpha.$$

The infimum of the numbers  $\alpha$  (if exists) for which this holds is defined to be the similarity degree of  $\mathcal{A}$ . We denote it by  $d(\mathcal{A})$ . If there is no such pair  $(\alpha, k)$ , we define  $d(\mathcal{A}) = \infty$ . It was shown in [18] that Kadison's Similarity Problem for a unital C\*-algebra  $\mathcal{A}$  has an affirmative answer if and only if  $d(\mathcal{A}) < \infty$ .

The following is a list of some recent results on the similarity degrees of infinite dimensional unital C\*-algebras. (We have no intention to make the list complete.)

- (i)  $d(\mathcal{A}) = 2$  if and only if  $\mathcal{A}$  is nuclear. ([2], [4], [21])
- (ii) If  $\mathcal{A} = B(H)$  for some Hilbert space  $H$ , then  $d(\mathcal{A}) = 3$ . ([9], [20])
- (iii) If  $\mathcal{A}$  is a type II<sub>1</sub> factor with Property  $\Gamma$ ,  $d(\mathcal{A}) \leq 5$ , ([20]). This result was later improved in [6] to  $d(\mathcal{A}) = 3$ .
- (iv) If  $\mathcal{A}$  is a minimal tensor product of two C\*-algebras, one of which is nuclear and contains matrices of any order, then  $d(\mathcal{A}) \leq 5$ . ([22])
- (v) If  $\mathcal{A}$  is  $\mathcal{Z}$ -stable, then  $d(\mathcal{A}) \leq 5$ . ([13])
- (vi) If every II<sub>1</sub> factor \*-representation of a separable C\*-algebra  $\mathcal{A}$  has Property  $\Gamma$ , then  $d(\mathcal{A}) \leq 11$ . ([11])

In the paper, we are interested in Kadison's Similarity Problem for type II<sub>1</sub> von Neumann algebras with Property  $\Gamma$ . Recall the definition of Property  $\Gamma$  for a type II<sub>1</sub> von Neumann algebra from [23]. Suppose  $\mathcal{M}$  is a type II<sub>1</sub> von Neumann algebra with a predual  $\mathcal{M}_\#$ . Suppose  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  is the weak-\* topology on  $\mathcal{M}$  induced from  $\mathcal{M}_\#$ . We say that  $\mathcal{M}$  has Property  $\Gamma$  if and only if  $\forall a_1, a_2, \dots, a_k \in \mathcal{M}$  and  $\forall n \in \mathbb{N}$ , there exist a partially ordered set  $\Lambda$  and a family of projections

$$\{p_{i\lambda} : 1 \leq i \leq n; \lambda \in \Lambda\} \subseteq \mathcal{M}$$

satisfying

- (i) For each  $\lambda \in \Lambda$ ,  $p_{1\lambda}, p_{2\lambda}, \dots, p_{n\lambda}$  are mutually orthogonal equivalent projections in  $\mathcal{M}$  with sum  $I$ .
- (ii) For each  $1 \leq i \leq n$  and  $1 \leq j \leq k$ ,

$$\lim_{\lambda} (p_{i\lambda} a_j - a_j p_{i\lambda})^* (p_{i\lambda} a_j - a_j p_{i\lambda}) = 0 \quad \text{in } \sigma(\mathcal{M}, \mathcal{M}_\#) \text{ topology.}$$

The first result we obtain in the paper is the following equivalent definition of Property  $\Gamma$  for a countably decomposable type II<sub>1</sub> von Neumann algebra, which gives an analogue of Murray and von Neumann's definition of Property  $\Gamma$  for a type II<sub>1</sub> factor.

**Proposition 3.5.** *Let  $\mathcal{M}$  be a countably decomposable type II<sub>1</sub> von Neumann algebra and  $\mathcal{Z}_{\mathcal{M}}$  be the center of  $\mathcal{M}$ . Suppose  $\tau$  is a center valued trace from  $\mathcal{M}$  to  $\mathcal{Z}_{\mathcal{M}}$  such that  $\tau(a) = a$  for all  $a \in \mathcal{Z}_{\mathcal{M}}$ . Then the following are equivalent.*

- (1)  $\mathcal{M}$  has Property  $\Gamma$ .
- (2) There exist a positive inter  $n_0 \geq 2$  and a faithful normal tracial state  $\rho$  on  $\mathcal{M}$  such that, for any  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ , there exists a family of orthogonal

equivalent projections  $p_1, \dots, p_{n_0}$  in  $\mathcal{M}$  with sum  $I$  satisfying  $\|p_i a_j - a_j p_i\|_2 < \epsilon$  for all  $i = 1, \dots, n_0$  and  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ .

- (3) *There exists a faithful normal tracial state  $\rho$  on  $\mathcal{M}$  such that, for any  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ , there exists a unitary  $u \in \mathcal{M}$  satisfying (i)  $\tau(u) = 0$  and (ii)  $\|ua_j - a_j u\|_2 < \epsilon$  for all  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ .*

Next we are able to obtain an upper bound for the similarity degree of a countably decomposable type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ , which extends Theorem 13 in [20].

**Theorem 4.4.** *If  $\mathcal{M}$  is a countably decomposable type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ , then  $d(\mathcal{M}) \leq 5$ .*

From Theorem 4.4, it follows that Kadison's Similarity Problem for a countably decomposable type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$  has an affirmative answer.

The last main result we obtained in the paper is the following computation of the similarity degree for a class of C\*-algebras.

**Theorem 5.3.** *Suppose  $\mathcal{A}$  is a separable unital C\*-algebra satisfying*

*Condition (G): if  $\pi$  is a unital \*-representation of  $\mathcal{A}$  on a Hilbert space  $H$  such that  $\pi(\mathcal{A})''$  is a type II<sub>1</sub> factor, then  $\pi(\mathcal{A})''$  has Property  $\Gamma$ , where  $\pi(\mathcal{A})''$  is the von Neumann algebra generated by  $\pi(\mathcal{A})$  in  $B(H)$ .*

*Then  $d(\mathcal{A}) \leq 3$ . Moreover, if  $\mathcal{A}$  is non-nuclear, then  $d(\mathcal{A}) = 3$ .*

As a corollary, we get that if  $\mathcal{A}$  is a minimal tensor product of two separable unital C\*-algebras, one of which is nuclear and has no finite dimensional \*-representations, then  $d(\mathcal{A}) \leq 3$ . In particular, the similarity degree of a  $\mathcal{Z}$ -stable, non-nuclear, separable, unital C\*-algebra is equal to 3. This gives a generalization of earlier results in [22], [13], [11].

The paper is organized as follows. In section 2, we introduce notation and preliminaries. In section 3, we will give an equivalent definition of Property  $\Gamma$  for countably decomposable type II<sub>1</sub> von Neumann algebras. In section 4, we show that if  $\mathcal{M}$  is a countably decomposable type II<sub>1</sub> von Neumann algebra Property  $\Gamma$ , then  $d(\mathcal{M}) \leq 5$ . By the result in Section 4, we prove in Section 5 that if the type II<sub>1</sub> central summand in the type decomposition of a von Neumann algebra  $\mathcal{M}$  is a countably decomposable von Neumann algebra with Property  $\Gamma$ , then any bounded,  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  to  $\sigma(B(H), B(H)_\#)$  continuous, unital homomorphism  $\phi : \mathcal{M} \rightarrow B(H)$  is completely bounded and  $\|\phi\|_{cb} \leq \|\phi\|^3$ . As a consequence of this result, we obtain that, if a separable unital C\*-algebra  $\mathcal{A}$  has Property c\*- $\Gamma$ , then  $d(\mathcal{A}) \leq 3$ . This is the first paper of our series. In our following work in [24], the class of separable unital C\*-algebras with Property c\*- $\Gamma$  will be applied to show that, if  $\mathcal{M}$  is a type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ , then  $d(\mathcal{M}) = 3$ .

## 2. Preliminaries

**2.1. Similarity length and similarity degree of a unital C\*-algebra.** In this subsection, we recall Pisier's similarity length and similarity degree for a unital C\*-algebra.

DEFINITION 2.1. ([19]) A unital operator algebra  $\mathcal{A}$  has finite similarity length at most  $l \in \mathbb{N}$  if there exists a constant  $C$  such that, for any  $k \in \mathbb{N}$  and any  $x \in M_k(\mathcal{A})$ , there exist an  $n \in \mathbb{N}$  and scalar matrices  $\alpha_0 \in M_{k,n}(\mathbb{C})$ ,  $\alpha_1 \in M_n(\mathbb{C})$ ,  $\dots$ ,  $\alpha_{l-1} \in M_n(\mathbb{C})$ ,  $\alpha_l \in M_{n,k}(\mathbb{C})$  and diagonal matrices  $D_1, D_2, \dots, D_l \in M_n(\mathcal{A})$  such that

$$x = \alpha_0 D_1 \alpha_1 D_2 \dots D_l \alpha_l$$

and

$$\left(\prod_{i=0}^l \|\alpha_i\|\right) \left(\prod_{i=1}^l \|D_i\|\right) \leq C \|x\|.$$

The length  $l(\mathcal{A})$  is defined to be the least possible  $l$  for which these conditions are fulfilled.

It was verified in [19] that the Kadison's Similarity Problem has a positive answer for a unital  $C^*$ -algebra if and only if the  $C^*$ -algebra has finite similarity length.

DEFINITION 2.2. ([18]) Let  $\mathcal{A}$  be a unital  $C^*$ -algebra. We define the similarity degree of  $\mathcal{A}$  to be the infimum of all positive numbers  $\alpha$  (if it exists) for which there is  $k > 0$  such that, for every bounded unital homomorphism  $\rho$  of  $\mathcal{A}$  on a Hilbert space  $H$ , we have

$$\|\rho\|_{cb} \leq k \|\rho\|^\alpha.$$

We denote such infimum by  $d(\mathcal{A})$ . If there are no such pairs  $(\alpha, k)$ , we define  $d(\mathcal{A}) = \infty$ .

It was proved in [18] that the similarity degree and the similarity length of a unital  $C^*$ -algebra (if they are finite) are the same integer.

**2.2. Dual space.** Suppose  $\mathcal{A}$  is a unital  $C^*$ -algebra. Its dual space, the set of all bounded linear functionals on  $\mathcal{A}$ , is denoted by  $\mathcal{A}^\#$ . The second dual space  $\mathcal{A}^{\#\#}$  of  $\mathcal{A}$  is isomorphic to  $\pi(\mathcal{A})''$ , where  $\pi$  is the universal representation of  $\mathcal{A}$  and  $\pi(\mathcal{A})''$  is the von Neumann algebra generated by  $\pi(\mathcal{A})$ . Thus  $\mathcal{A}^{\#\#}$  is always viewed as a von Neumann algebra and  $\mathcal{A}$  becomes a  $C^*$ -subalgebra of  $\mathcal{A}^{\#\#}$  when  $\mathcal{A}$  is canonically embedded into  $\mathcal{A}^{\#\#}$ .

Suppose  $\mathcal{M}$  is a von Neumann algebra with a (unique) predual space  $\mathcal{M}_\#$ . We will denote by  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  the weak-\* topology on  $\mathcal{M}$  induced by  $\mathcal{M}_\#$ .

The following lemma is well-known. (See Theorem 1 in [1] or Proposition 1.21.13 in [25])

LEMMA 2.3. Suppose  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $\phi : \mathcal{A} \rightarrow B(H)$  is a bounded unital homomorphism of  $\mathcal{A}$  on a Hilbert space  $H$ . Then  $\phi$  can be extended to a bounded unital homomorphism  $\bar{\phi} : \mathcal{A}^{\#\#} \rightarrow B(H)$  that is  $\sigma(\mathcal{A}^{\#\#}, \mathcal{A}^\#) \rightarrow \sigma(B(H), B(H)_\#)$  continuous (in other words, it is weak-\* to weak-\* continuous), where  $B(H)_\#$  is the predual of  $B(H)$ . Moreover  $\|\bar{\phi}\| = \|\phi\|$ .

**2.3. Direct integral.** The concept of direct integral was introduced by von Neumann in [26]. Here are some results about direct integral that we need in this paper.

LEMMA 2.4. ([16]) Suppose  $\mathcal{M}$  is a von Neumann algebra acting on a separable Hilbert space  $H$ . Let  $\mathcal{Z}_\mathcal{M}$  be the center of  $\mathcal{M}$ . Then there is a direct integral decomposition of  $\mathcal{M}$  relative to  $\mathcal{Z}_\mathcal{M}$ , i.e. there exists a locally compact complete separable metric measure space  $(X, \mu)$  such that

- (i)  $H$  is (unitarily equivalent to) the direct integral of  $\{H_s : s \in X\}$  over  $(X, \mu)$ , where each  $H_s$  is a separable Hilbert space,  $s \in X$ .
- (ii)  $\mathcal{M}$  is (unitarily equivalent to) the direct integral of  $\{\mathcal{M}_s : s \in X\}$  over  $(X, \mu)$ , where  $\mathcal{M}_s$  is a factor in  $B(H_s)$  almost everywhere. Also, if  $\mathcal{M}$  is of type  $I_n$  ( $n$  could be infinite),  $II_1$ ,  $II_\infty$  or  $III$ , then the components  $\mathcal{M}_s$  are, almost everywhere, of type  $I_n$ ,  $II_1$ ,  $II_\infty$  or  $III$ , respectively.

Moreover, the center  $\mathcal{Z}_{\mathcal{M}}$  is (unitarily equivalent to) the algebra of diagonalizable operators relative to this decomposition.

LEMMA 2.5. ([16]) If  $H$  is the direct integral of  $\{H_s\}$  over  $(X, \mu)$ ,  $\mathcal{M}$  is a decomposable von Neumann algebra on  $H$  (i.e every operator in  $\mathcal{M}$  is decomposable relative to the direct integral decomposition, see Definition 14.1.6 in [16]) and  $\omega$  is a normal state on  $\mathcal{M}$ , then there is a mapping,  $s \rightarrow \omega_s$ , where  $\omega_s$  is a positive normal linear functional on  $\mathcal{M}_s$ , and  $\omega(a) = \int_X \omega_s(a(s)) d\mu$  for each  $a$  in  $\mathcal{M}$ . If  $\mathcal{M}$  contains the algebra  $\mathcal{C}$  of diagonalizable operators and  $\omega|_{EME}$  is faithful or tracial, for some projection  $E$  in  $\mathcal{M}$ , then  $\omega_s|_{E(s)\mathcal{M}_sE(s)}$  is, accordingly, faithful or tracial almost everywhere.

REMARK 2.6. Let  $\mathcal{M} = \int_X \bigoplus \mathcal{M}_s d\mu$  and  $H = \int_X \bigoplus H_s d\mu$  be the direct integral decompositions of  $\mathcal{M}$  and  $H$  relative to the center of  $\mathcal{M}$ . By the argument in section 14.1 in [16], we can find a separable Hilbert space  $K$  and a family of unitaries  $\{U_s : H_s \rightarrow K : s \in X\}$  such that  $s \rightarrow U_s x(s)$  is measurable (i.e.  $s \rightarrow \langle U_s x(s), y \rangle$  is measurable for any vector  $y$  in  $K$ ) for every  $x \in H$  and  $s \rightarrow U_s a(s) U_s^*$  is measurable (i.e.  $s \rightarrow \langle U_s a(s) U_s^* y, z \rangle$  is measurable for any vectors  $y, z$  in  $K$ ) for every decomposable operator  $a \in B(H)$ .

The following corollary follows directly from Lemma 14.1.17 and Lemma 14.1.15 in [16].

LEMMA 2.7. Let  $\mathcal{M}$  be a von Neumann algebra acting on a separable Hilbert space  $H$ . Let  $\mathcal{M} = \int_X \bigoplus \mathcal{M}_s d\mu$  and  $H = \int_X \bigoplus H_s d\mu$  be the direct integral decompositions of  $\mathcal{M}$  and  $H$  as in Lemma 2.4. There exists a SOT dense sequence  $\{a_j : j \in \mathbb{N}\}$  in the unit ball  $(\mathcal{M})_1$  of  $\mathcal{M}$  (or dense in the unit ball  $(\mathcal{M}')_1$  of  $\mathcal{M}$ ) such that the sequence  $\{a_j(s) : j \in \mathbb{N}\}$  is SOT dense in the unit ball  $(\mathcal{M}_s)_1$  (or  $(\mathcal{M}'_s)_1$ , respectively) for almost every  $s \in X$ .

### 3. Type $II_1$ von Neumann algebras with Property $\Gamma$

Property  $\Gamma$  of a type  $II_1$  factor  $\mathcal{A}$  was introduced by Murray and von Neumann in [17]. Suppose  $\mathcal{A}$  is a type  $II_1$  factor with a trace  $\tau$ . Then  $\mathcal{A}$  has Property  $\Gamma$  if and only if, given  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{A}$ , there exists a unitary  $u \in \mathcal{A}$ , such that

- (i)  $\tau(u) = 0$ ;
- (ii)  $\|ua_j - a_j u\|_2 < \epsilon, 1 \leq j \leq k$ .

An equivalent definition of Property  $\Gamma$  for a type  $II_1$  factor was given by Dixmier in [8]. Suppose  $\mathcal{A}$  is a type  $II_1$  factor with a trace  $\tau$ . It has Property  $\Gamma$  if and only if, given  $\epsilon > 0$ , elements  $a_1, a_2, \dots, a_k \in \mathcal{A}$  and a positive integer  $n$ , there exist orthogonal equivalent projections  $\{p_i : 1 \leq i \leq n\} \subset \mathcal{A}$  with sum  $I$ , such that

$$\|p_i a_j - a_j p_i\|_2 < \epsilon, 1 \leq i \leq n, 1 \leq j \leq k.$$

The following definition of Property  $\Gamma$  for a type  $II_1$  von Neumann algebra was given in [23]:

**DEFINITION 3.1.** ([23]) *Suppose  $\mathcal{M}$  is a type  $II_1$  von Neumann algebra with a predual  $\mathcal{M}_\#$ . Suppose that  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  is the weak-\* topology on  $\mathcal{M}$  induced from  $\mathcal{M}_\#$ . We say that  $\mathcal{M}$  has Property  $\Gamma$  if and only if  $\forall a_1, a_2, \dots, a_k \in \mathcal{M}$  and  $\forall n \in \mathbb{N}$ , there exist a partially ordered set  $\Lambda$  and a family of projections*

$$\{p_{i\lambda} : 1 \leq i \leq n; \lambda \in \Lambda\} \subseteq \mathcal{M}$$

*satisfying*

- (i) *For each  $\lambda \in \Lambda$ ,  $\{p_{1\lambda}, p_{2\lambda}, \dots, p_{n\lambda}\}$  is a family of orthogonal equivalent projections in  $\mathcal{M}$  with sum  $I$ .*
- (ii) *For each  $1 \leq i \leq n$  and  $1 \leq j \leq k$ ,*

$$\lim_{\lambda} (p_{i\lambda} a_j - a_j p_{i\lambda})^* (p_{i\lambda} a_j - a_j p_{i\lambda}) = 0 \quad \text{in } \sigma(\mathcal{M}, \mathcal{M}_\#) \text{ topology.}$$

It was proved in [23] that Definition 3.1 coincides with Dixmier's (also with Murray and von Neumann's) definition of Property  $\Gamma$  for a type  $II_1$  factor.

**EXAMPLE 3.2.** *Let  $\mathcal{M}_1$  be a type  $II_1$  factor and  $\mathcal{M}_2$  a type  $II_1$  von Neumann algebra. Suppose  $\mathcal{M}_1$  has Property  $\Gamma$  (e.g. hyperfinite type  $II_1$  factor). Then  $\mathcal{M}_1 \otimes \mathcal{M}_2$  has Property  $\Gamma$ .*

Let  $\mathcal{M}$  be a type  $II_1$  von Neumann algebra acting on a separable Hilbert space  $H$  and  $\mathcal{Z}_{\mathcal{M}}$  be the center of  $\mathcal{M}$ . Let  $\mathcal{M} = \int_X \oplus \mathcal{M}_s d\mu$  and  $H = \int_X \oplus H_s d\mu$  be the direct integral decompositions of  $\mathcal{M}$  and  $H$  over  $(X, \mu)$  relative to  $\mathcal{Z}_{\mathcal{M}}$ . By Proposition 3.12 in [23],  $\mathcal{M}$  has Property  $\Gamma$  if and only if  $\mathcal{M}_s$  has Property  $\Gamma$  for almost every  $s \in X$ .

The following proposition gives an equivalent definition of Property  $\Gamma$  for a type  $II_1$  von Neumann algebra with separable predual, as an analogous to Murray and von Neumann's definition for type  $II_1$  factors.

**PROPOSITION 3.3.** *Let  $\mathcal{M}$  be a type  $II_1$  von Neumann algebra with separable predual and  $\mathcal{Z}_{\mathcal{M}}$  be the center of  $\mathcal{M}$ . Suppose  $\tau$  is a center valued trace from  $\mathcal{M}$  to  $\mathcal{Z}_{\mathcal{M}}$  such that  $\tau(a) = a$  for all  $a \in \mathcal{Z}_{\mathcal{M}}$ . Then  $\mathcal{M}$  has Property  $\Gamma$  if and only if there exists a faithful normal tracial state  $\rho$  on  $\mathcal{M}$  such that, for any  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ , there exists a unitary  $u \in \mathcal{M}$  satisfying (i)  $\tau(u) = 0$  and (ii)  $\|ua_j - a_j u\|_2 < \epsilon$  for all  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ .*

**PROOF.** (Part I) Suppose  $\mathcal{M}$  has Property  $\Gamma$ . Fix  $n \geq 2$ ,  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ . Then by Corollary 3.4 in [23], there exist a faithful normal tracial state  $\rho$  and  $n$  equivalent orthogonal projections  $p_1, p_2, \dots, p_n$  with sum  $I$  such that  $\|p_i a_j - a_j p_i\|_2 < \epsilon/n$  for all  $i = 1, 2, \dots, n$  and  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ . Let  $u = p_1 + \lambda p_2 + \dots + \lambda^{n-1} p_n$ , where  $\lambda = e^{2\pi i/n}$  is the  $n$ -th root of unit. Then  $u$  is a unitary in  $\mathcal{M}$  satisfying  $\|ua_j - a_j u\|_2 < \epsilon$  for all  $j = 1, 2, \dots, k$ . Since  $p_1, p_2, \dots, p_n$  are equivalent projections,  $\tau(p_1) = \tau(p_2) = \dots = \tau(p_n)$  and thus  $\tau(u) = (1 + \lambda + \dots + \lambda^{n-1})\tau(p_1) = 0$ .

(Part II) Conversely, suppose that there exists a faithful normal tracial state  $\rho$  such that, for any  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ , there exists a unitary  $u \in \mathcal{M}$  satisfying (i)  $\tau(u) = 0$

and (ii)  $\|ua_j - a_ju\|_2 < \epsilon$  for all  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ . From the fact that  $\mathcal{M}$  has separable predual, it follows  $\mathcal{M}$  is countably generated. Thus, from the preceding argument, we know there exists a sequence  $\{u_i : i \in \mathbb{N}\}$  of unitaries in  $\mathcal{M}$  satisfying

- (1)  $\tau(u_i) = 0$  for each  $i \in \mathbb{N}$ ;
- (2)

$$\lim_{i \rightarrow \infty} \|u_i a - a u_i\|_2 = 0 \quad \text{for each } a \in \mathcal{M}. \quad (3.1)$$

Since  $\mathcal{M}$  has separable predual, by Propostion A.2.1 in [14], there is a faithful normal representation  $\pi$  of  $\mathcal{M}$  on the separable Hilbert space. Replacing  $\mathcal{M}$  by  $\pi(\mathcal{M})$  in the following if necessary, we assume that  $\mathcal{M}$  acts on a separable Hilbert space  $H$ .

By Lemma 2.4, relative to  $\mathcal{Z}_{\mathcal{M}}$ , we obtain a direct integral decomposition  $\mathcal{M} = \int_X \bigoplus M_s d\mu$  over  $(X, \mu)$  and we may assume that  $\mathcal{M}_s$  is a type II<sub>1</sub> factor with a trace  $\tau_s$  for every  $s \in X$ .

Since  $\rho$  is a faithful normal tracial state, by Lemma 2.5, we can further assume that there is a positive faithful normal tracial functional  $\rho_s$  on  $\mathcal{M}_s$  for every  $s \in X$  such that

$$\rho(a) = \int_X \rho_s(a(s)) d\mu \quad \text{for each } a \in \mathcal{M}..$$

Therefore  $\rho_s$  is a positive multiple of the trace  $\tau_s$  on  $\mathcal{M}_s$  for every  $s \in X$ .

By Lemma 2.7, we may assume  $\{b_j : j \in \mathbb{N}\}$  is a *SOT* dense subset of the unit ball  $(\mathcal{M})_1$  of  $\mathcal{M}$  such that that  $\{b_j(s) : j \in \mathbb{N}\}$  is *SOT* dense in the unit ball of  $\mathcal{M}_s$  for every  $s \in X$ .

Let  $u_i^{(0)} = u_i$  for each  $i \in \mathbb{N}$ . In the following we will construct a family of unitaries  $\{u_i^{(k)} : i, k \in \mathbb{N}\}$  and a family of  $\mu$ -null subsets  $\{X_k : k \in \mathbb{N}\}$  of  $X$  such that, for each  $k \in \mathbb{N}$ ,

- (i')  $\{u_i^{(k)} : i \in \mathbb{N}\}$  is a subsequence of  $\{u_i^{(k-1)} : i \in \mathbb{N}\}$ ;
- (ii')  $\lim_{i \rightarrow \infty} \rho_s((u_i^{(k)}(s)b_k(s) - a_k(s)u_i^{(k)}(s))^*(u_i^{(k)}(s)a_k(s) - a_k(s)u_i^{(k)}(s))) = 0$  for any  $s \in X \setminus X_k$ .

By (3.1), we get

$$\begin{aligned} & \lim_{i \rightarrow \infty} \|u_i a - a u_i\|_2^2 \\ &= \lim_{i \rightarrow \infty} \int_X \rho_s((u_i(s)b_1(s) - b_1(s)u_i(s))^*(u_i(s)b_1(s) - b_1(s)u_i(s))) d\mu \\ &= 0. \end{aligned}$$

Therefore there exists a  $\mu$ -null subset  $X_1$  of  $X$  and a subsequence  $\{u_i^{(1)}\}$  of  $\{u_i^{(0)}\}$  such that, for any  $s \in X \setminus X_1$ ,

$$\lim_{i \rightarrow \infty} \rho_s((u_i^{(1)}(s)b_1(s) - b_1(s)u_i^{(1)}(s))^*(u_i^{(1)}(s)b_1(s) - b_1(s)u_i^{(1)}(s))) = 0. \quad (3.2)$$

Since  $\rho_s$  is a positive multiple of  $\tau_s$  for every  $s \in X$ , (3.2) gives

$$\lim_{i \rightarrow \infty} \|u_i^{(1)}(s)b_1(s) - b_1(s)u_i^{(1)}(s)\|_{2,s} = 0,$$

where  $\|\cdot\|_{2,s}$  is the 2-norm induced by  $\tau_s$  on  $\mathcal{M}_s$ . Again, there exists a  $\mu$ -null subset  $X_2$  of  $X$  and a subsequence  $\{u_i^{(2)}\}$  of  $\{u_i^{(1)}\}$  such that, for any  $s \in X \setminus X_2$ ,

$$\lim_{i \rightarrow \infty} \|u_i^{(2)}(s)b_2(s) - b_2(s)u_i^{(2)}(s)\|_{2,s} = 0.$$

Continuing in this way, we obtain a  $\mu$ -null subset  $X_k$  of  $X$  and a subsequence  $\{u_i^{(k)}\}$  of  $\{u_i^{(k-1)}\}$  for each  $k \geq 1$  such that, for any  $s \in X \setminus X_k$ ,

$$\lim_{i \rightarrow \infty} \|u_i^{(k)}(s)b_k(s) - b_k(s)u_i^{(k)}(s)\|_{2,s} = 0.$$

The argument in the preceding paragraph produces a subsequence  $\{u_i^{(i)}\}$  of  $\{u_i\}$  such that

$$\lim_{i \rightarrow \infty} \|u_i^{(i)}(s)b_j(s) - b_j(s)u_i^{(i)}(s)\|_{2,s} = 0$$

for any  $j \in \mathbb{N}, s \in X \setminus X_0$ , where  $X_0 = \cup_{j \in \mathbb{N}} X_j$  is a  $\mu$ -null subset of  $X$ . Replacing  $\{u_i : i \in \mathbb{N}\}$  by  $\{u_i^{(i)}\}$  and  $X$  by  $X \setminus X_0$  if necessary, we might assume

$$\lim_{i \rightarrow \infty} \|u_i(s)b_j(s) - b_j(s)u_i(s)\|_{2,s} = 0 \quad \text{for any } j \in \mathbb{N}, s \in X.$$

Since  $\{b_j(s) : j \in \mathbb{N}\}$  is SOT dense in the unit ball of  $\mathcal{M}_s$ , we get

$$\lim_{i \rightarrow \infty} \|u_i(s)a - au_i(s)\|_{2,s} = 0 \quad \text{for any } a \in \mathcal{M}_s, s \in X. \quad (3.3)$$

For each  $i \geq 1$ , since  $u_i$  is a unitary in  $\mathcal{M}$ , there exists a  $\mu$ -null subset  $Y_i$  of  $X$  such that  $u_i(s)$  is a unitary in  $\mathcal{M}_s$  for each  $s \in X \setminus Y_i$ . Let  $Y_0 = \cup_{i=1}^{\infty} Y_i$ . Then  $\mu(Y_0) = 0$  and  $u_i(s)$  is a unitary in  $\mathcal{M}_s$  for all  $i \in \mathbb{N}, s \in X \setminus Y_0$ . So we may just assume that

$$u_i(s) \text{ is a unitary in } \mathcal{M}_s \text{ for any } i \in \mathbb{N}, s \in X. \quad (3.4)$$

For each  $i \in \mathbb{N}$ , from the Dixmier Approximation Theorem and the fact that  $\tau(u_i) = 0$ , 0 is in the norm closure of the convex hull of  $\{v^*u_iv : v \text{ is a unitary in } \mathcal{M}\}$ . Therefore there exist a sequence of positive integers  $\{k_n \in \mathbb{N}\}$ , a family of positive numbers  $\{\lambda_j^{(n)} : 1 \leq j \leq k_n, n \in \mathbb{N}\} \subseteq [0, 1]$  and a family of unitaries  $\{v_j^{(n)} : 1 \leq j \leq k_n\}$  in  $\mathcal{M}$  such that

$$\sum_{j=1}^{k_n} \lambda_j^{(n)} = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \left\| \sum_{j=1}^{k_n} \lambda_j^{(n)} (v_j^{(n)})^* u_i v_j^{(n)} \right\| = 0.$$

By Proposition 14.1.9 in [16],  $\|a\|$  is the essential bound of  $\{\|a(s)\| : s \in X\}$  for any  $a \in \mathcal{M}$ . Note that  $\{v_j^{(n)} : 1 \leq j \leq k_n\}$  is a family of unitaries in  $\mathcal{M}$ . We know that there exists a  $\mu$ -null subset  $Z_0$  of  $X$  such that

- (a) for each  $n \in \mathbb{N}$ , each  $1 \leq j \leq k_n$  and each  $s \in X \setminus Z_0$ ,  $v_j^n(s)$  is a unitary in  $\mathcal{M}_s$ ;
- (b) for each  $i \in \mathbb{N}$  and each  $s \in X \setminus Z_0$ ,

$$\lim_{n \rightarrow \infty} \left\| \sum_{j=1}^{k_n} \lambda_j^{(n)} (v_j^{(n)}(s))^* u_i(s) v_j^{(n)}(s) \right\| = 0.$$



Now from (a), (b) and the Dixmier Approximation Theorem, we obtain that

$$\tau_s(u_i(s)) = 0, \quad \text{for any } i \in \mathbb{N}, s \in X \setminus Z_0. \quad (3.5)$$

Here  $Z_0$  is a  $\mu$ -null subset of  $X$ .

By (3.5), (3.4), and (3.3),  $\mathcal{M}_s$  is a type  $II_1$  factor with Property  $\Gamma$  for each  $s \in X \setminus Z_0$ . Therefore by Proposition 3.12 in [23],  $\mathcal{M}$  has Property  $\Gamma$ .  $\square$

**LEMMA 3.4.** *Let  $\mathcal{M}$  be a countably decomposable type  $II_1$  von Neumann algebra and  $\mathcal{Z}_{\mathcal{M}}$  be the center of  $\mathcal{M}$ . Suppose  $\tau$  is a center valued trace from  $\mathcal{M}$  to  $\mathcal{Z}_{\mathcal{M}}$  such that  $\tau(a) = a$  for all  $a \in \mathcal{Z}_{\mathcal{M}}$ . Suppose there exists a faithful normal tracial state  $\rho$  on  $\mathcal{M}$  such that,*

*for any  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ , there exists a unitary  $u \in \mathcal{M}$  satisfying*  
*(i)  $\tau(u) = 0$  and (ii)  $\|ua_j - a_ju\|_2 < \epsilon$  for all  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ .*

*Then, for any finite subset  $F$  of  $\mathcal{M}$ , there exists a von Neumann subalgebra  $\mathcal{M}_1$  of  $\mathcal{M}$  such that*

- (1)  $\mathcal{M}_1$  has separable predual.
- (2)  $F \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ .
- (3)  $\mathcal{M}_1$  is a type  $II_1$  von Neumann algebra with Property  $\Gamma$  in the sense of Definition 3.1.

**PROOF.** We are going to prove the following claim first.

**Claim 3.4.1.** *For any finite subset  $K$  of  $\mathcal{M}$  and any  $\epsilon > 0$ , there exist an  $n \in \mathbb{N}$ , unitary elements  $u, v_1, \dots, v_n \in \mathcal{M}$ , such that  $\tau(u) = 0$ ;  $\forall x \in K$ ,  $\|ux - xu\|_2 < \epsilon$ ; and  $\forall x \in K$ , there is an element  $y$  in the convex hull of  $\{v_1xv_1^*, \dots, v_nxv_n^*\}$  satisfying  $\|y - \tau(x)\| < \epsilon$ .*

**Proof of Claim 3.4.1:** From the assumption on the faithful normal tracial state  $\rho$  of  $\mathcal{M}$ , for a finite subset  $K$  of  $\mathcal{M}$  and an  $\epsilon > 0$ , there is a unitary  $u$  in  $\mathcal{M}$  such that  $\tau(u) = 0$  and  $\|ux - xu\|_2 < \epsilon$  for all  $x \in K$ . By Dixmier Approximation Theorem,  $\tau(x) \in \text{conv}_{\mathcal{M}}(x)^\perp$  for all  $x \in \mathcal{M}$ . Therefore, there exist a positive integer  $n$  and unitary elements  $v_1, \dots, v_n \in \mathcal{M}$  such that for any  $x \in K$ , there is an element  $y$  in the convex hull of  $\{v_1xv_1^*, \dots, v_nxv_n^*\}$  satisfying  $\|y - \tau(x)\| < \epsilon$ . This finished the proof of the claim.

(Continue the proof of Lemma 3.4) Let  $F$  be a finite subset of  $\mathcal{M}$ .

Let  $F_1 = F$  and  $t = 1$ . From Claim 3.4.1, there exist a positive integer  $n_1$ , unitary elements  $u_1, v_1^{(1)}, \dots, v_{n_1}^{(1)} \in \mathcal{M}$ , such that  $\tau(u_1) = 0$ ;  $\|u_1x - xu_1\|_2 < 1$ ,  $\forall x \in F_1$ ; and for any  $x \in F_1$ , there is a  $y$  in the convex hull of  $\{v_1^{(1)}x(v_1^{(1)})^*, \dots, v_{n_1}^{(1)}x(v_{n_1}^{(1)})^*\}$  satisfying  $\|y - \tau(x)\| < 1$ . Let  $F_2 = F_1 \cup \{u_1, v_1^{(1)}, \dots, v_{n_1}^{(1)}\}$ .

Assume that  $F_1 \subseteq F_2 \subseteq \dots \subseteq F_t$  have been constructed for some  $t \geq 2$ . Again, from Claim 3.4.1, there exist a positive integer  $n_t$ , unitary elements  $u_t, v_1^{(t)}, \dots, v_{n_t}^{(t)} \in \mathcal{M}$ , such that  $\tau(u_t) = 0$ ;  $\|u_tx - xu_t\|_2 < 1/t$ ,  $\forall x \in F_t$ ; and for any  $x \in F_t$ , there is a  $y$  in the convex hull of  $\{v_1^{(t)}x(v_1^{(t)})^*, \dots, v_{n_t}^{(t)}x(v_{n_t}^{(t)})^*\}$  satisfying  $\|y - \tau(x)\| < 1/t$ . Let  $F_{t+1} = F_t \cup \{u_t, v_1^{(t)}, \dots, v_{n_t}^{(t)}\}$ .

Continuing the preceding process, we are able to obtain an increasing sequence  $\{F_t\}_{t=1}^\infty$  of finite subsets of  $\mathcal{M}$  and a sequence of unitaries  $\{u_t\}_{t=1}^\infty$  of  $\mathcal{M}$  such that,  $\forall t \geq 1$ ,

- (0)  $u_t \in F_{t+1} \subseteq \mathcal{M}$ ;
- (i)  $\tau(u_t) = 0$ ;

- (ii)  $\|u_t x - x u_t\|_2 < 1/t$ ,  $\forall x \in F_t$ ;
- (iii) for  $1 \leq i \leq t$ , there is  $y_i$  in the convex hull of  $\{v u_i v^* : v \text{ is a unitary element in } F_{t+2}\}$  satisfying  $\|y_i - \tau(u_i)\| = \|y_i\| < 1/(t+1)$ ;

Let  $\mathcal{M}_1$  be the von Neumann subalgebra generated by  $\{F_t : t \geq 1\}$  in  $\mathcal{M}$  and  $\mathcal{Z}_1$  be the center of  $\mathcal{M}_1$ . Suppose  $\tau_1$  be a center-value trace on  $\mathcal{M}_1$  such that  $\tau_1(a) = a$ ,  $\forall a \in \mathcal{Z}_1$ . Then  $\rho$  is still a faithful normal tracial state of  $\mathcal{M}_1$ . Since  $\mathcal{M}_1$  is countably generated, we know that (1)  $\mathcal{M}_1$  has a separable predual. Obviously, (2)  $F \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ .

We claim that (3)  $\mathcal{M}_1$  is a von Neumann algebra with Property  $\Gamma$  in the sense of Definition 3.1. Notice  $\{F_t\}$  is an increasing sequence of subsets. We have, for each  $1 \leq t_1 < t < t_2 - 2$ ,

- (ii<sub>2</sub>)  $\|u_t x - x u_t\|_2 < 1/t$ ,  $\forall x \in F_{t_1}$ ;
- (iii<sub>2</sub>) there is  $y$  in the convex hull of  $\{v u_t v^* : v \text{ is a unitary in } F_{t_2}\}$  satisfying  $\|y\| < 1/t_2$ .

From (iii<sub>2</sub>), it induces by the Dixmier Approximation Theorem that

- (iii<sub>3</sub>)  $\tau_1(u_t) = 0$  for each  $t \geq 1$ .

From the existence of such sequence  $\{u_t\}_{t=1}^\infty$  in  $\mathcal{M}_1$  satisfying (iii<sub>3</sub>) and (ii<sub>2</sub>), it follows that  $\mathcal{M}_1$  is a type II<sub>1</sub> von Neumann algebra. From Proposition 3.3, we conclude that  $\mathcal{M}_1$  has Property  $\Gamma$ . The proof of the lemma is finished.  $\square$

Now we can quickly prove the following result.

**PROPOSITION 3.5.** *Let  $\mathcal{M}$  be a countably decomposable type II<sub>1</sub> von Neumann algebra and  $\mathcal{Z}_{\mathcal{M}}$  be the center of  $\mathcal{M}$ . Suppose  $\tau$  is a center valued trace from  $\mathcal{M}$  to  $\mathcal{Z}_{\mathcal{M}}$  such that  $\tau(a) = a$  for all  $a \in \mathcal{Z}_{\mathcal{M}}$ . Then the following are equivalent.*

- (1)  $\mathcal{M}$  has Property  $\Gamma$ .
- (2) *There exist a positive inter  $n_0 \geq 2$  and a faithful normal tracial state  $\rho$  on  $\mathcal{M}$  such that, for any  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ , there exists a family of orthogonal equivalent projections  $p_1, \dots, p_{n_0}$  in  $\mathcal{M}$  with sum  $I$  satisfying  $\|p_i a_j - a_j p_i\|_2 < \epsilon$  for all  $i = 1, \dots, n_0$  and  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ .*
- (3) *There exists a faithful normal tracial state  $\rho$  on  $\mathcal{M}$  such that, for any  $\epsilon > 0$  and elements  $a_1, a_2, \dots, a_k \in \mathcal{M}$ , there exists a unitary  $u \in \mathcal{M}$  satisfying (i)  $\tau(u) = 0$  and (ii)  $\|u a_j - a_j u\|_2 < \epsilon$  for all  $j = 1, 2, \dots, k$ , where  $\|\cdot\|_2$  is the 2-norm induced by  $\rho$ .*

**PROOF.** (1)  $\Rightarrow$  (2): It is clear.

(2)  $\Rightarrow$  (3): It follows from the similar arguments in Part I of the proof of Proposition 3.3.

(3)  $\Rightarrow$  (1): It follows from Proposition 3.3, Lemma 3.4 and Corollary 3.4 in [23].  $\square$

The following lemma will be needed in the next section.

**LEMMA 3.6.** *Let  $\mathcal{M}$  be a countably decomposable von Neumann algebra with Property  $\Gamma$ . For any finite subset  $F$  of  $\mathcal{M}$ , there exists a type II<sub>1</sub> von Neumann algebra  $\mathcal{M}_1$  with separable predual and with Property  $\Gamma$  such that  $F \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ .*

**PROOF.** It now follows from Proposition 3.5 and Lemma 3.4.  $\square$

REMARK 3.7. *It was shown in [23] that the cohomology group  $H^k(\mathcal{M}, \mathcal{M})$  of a type  $II_1$  von Neumann algebra with separable predual and Property  $\Gamma$  is trivial for any  $k \geq 2$ . By a similar argument as section 7 in [7], it follows from Theorem 6.4 in [23] and Lemma 3.6 that  $H^k(\mathcal{M}, \mathcal{M}) = 0$ ,  $k \geq 2$  for any countably decomposable type  $II_1$  von Neumann algebra with Property  $\Gamma$ .*

#### 4. Similarity degree of type $II_1$ von Neumann algebras with Property $\Gamma$

Let us recall the definition of similarity length of a unital C\*-algebra as given in [20].

DEFINITION 4.1. ([20]) *Let  $\mathcal{A}$  be a unital C\*-algebra. Fix  $n \in \mathbb{N}$ . For each  $x \in M_n(\mathcal{A})$  and  $d \in \mathbb{N}$ , we denote*

$$\|x\|_{(d, \mathcal{A})} = \inf \left\{ \prod_{i=1}^d \|\alpha_i\| \prod_{i=0}^d \|D_i\| \right\},$$

where the infimum runs over all possible representations  $x = \alpha_0 D_1 \alpha_1 D_2 \dots D_d \alpha_d$ ,  $\alpha_0 \in M_{k,n}(\mathbb{C})$ ,  $\alpha_1 \in M_n(\mathbb{C}), \dots, \alpha_{d-1} \in M_n(\mathbb{C}), \alpha_d \in M_{n,k}(\mathbb{C})$  are scalar matrices and  $D_1, D_2, \dots, D_d \in M_n(\mathcal{A})$  are diagonal matrices.

It is clear that, for any  $x \in M_n(\mathcal{A}), d \in \mathbb{N}$ ,

$$\|x\| \leq \|x\|_{(d, \mathcal{A})}$$

and

$$\|x\|_{(d+1, \mathcal{A})} \leq \|x\|_{(d, \mathcal{A})}.$$

It was shown in [20] that  $\|x\|_{(1, \mathcal{A})} \leq n\|x\|$  for any  $x \in M_n(\mathcal{A})$ .

Before we prove the main theorem of this section, we need the following lemmas.

LEMMA 4.2. *Let  $\mathcal{M}$  be a von Neumann algebra acting on a separable Hilbert space  $H$ . Let  $\mathcal{M} = \int_X \bigoplus \mathcal{M}_s d\mu$  and  $H = \int_X H_s d\mu$  be the direct integral decompositions of  $\mathcal{M}$  and  $H$  as in Lemma 2.4. Suppose  $p$  is a projection in  $\mathcal{M}$  such that there exist a  $\mu$ -null subset  $X_0$  of  $X$  and projections  $\{p_{i,s} \in \mathcal{M}_s : 2 \leq i \leq n, s \in X\}$  satisfying  $\{p(s), p_{2,s}, \dots, p_{n,s}\}$  is a family of  $n$  orthogonal equivalent projections in  $\mathcal{M}_s$  for all  $s \in X \setminus X_0$ . Then there exist  $n-1$  projections  $p_2, \dots, p_n$  in  $\mathcal{M}$  such that  $\{p, p_2, \dots, p_n\}$  is a set of  $n$  orthogonal equivalent projections in  $\mathcal{M}$ .*

PROOF. We let  $X_0$  be a  $\mu$ -null subset of  $X$  and  $\{p_{i,s} \in \mathcal{M}_s : 2 \leq i \leq n, s \in X\}$  be projections satisfying  $p(s), p_{2,s}, \dots, p_{n,s}$  are  $n$  orthogonal equivalent projections in  $\mathcal{M}_s$  for all  $s \in X \setminus X_0$ .

Let  $K$  be a separable Hilbert space and  $\{U_s : H_s \rightarrow K\}$  be a family of unitaries as in Remark 2.6. Denote by  $\mathcal{B}$  the unit ball of  $B(K)$  equipped with the  $*$ -strong operator topology. Then  $\mathcal{B}$  is metrizable by setting

$$d(S, T) = \sum_{j \in \mathbb{N}} 2^{-j} (\|(S - T)e_j\| + \|(S^* - T^*)e_j\|)$$

for any  $S, T \in \mathcal{B}$ , where  $\{e_j : j \in \mathbb{N}\}$  is an orthonormal basis for  $K$ . The metric space  $(\mathcal{B}, d)$  is complete and separable. For each  $i, j \in \{1, 2, \dots, n\}$ , let  $\mathcal{B}_{ij} = \mathcal{B}$ . Let  $\mathcal{C} = \prod_{i,j=1}^n \mathcal{B}_{ij}$  provided

with the product topology of the  $*$ -strong operator topology on each  $\mathcal{B}_{ij}$ . It follows that  $\mathcal{C}$  is metrizable and it is a complete separable metric space.

By Lemma 2.7, suppose  $\{a'_r : r \in \mathbb{N}\}$  is a strong operator dense sequence in the unit ball  $(\mathcal{M}')_1$  of  $\mathcal{M}'$  such that the sequence  $\{a'_r(s) : r \in \mathbb{N}\}$  is strong operator dense in the unit ball  $(\mathcal{M}'_s)_1$  of  $\mathcal{M}'_s$  for every  $s \in X$ .

We will denote by  $(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn}))$  an element in  $X \times \mathcal{C}$ . It is easy to see that the maps

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \rightarrow E_{11}, \quad (4.1)$$

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \rightarrow U_s p(s) U_s^*, \quad (4.2)$$

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \rightarrow E_{ij}, \quad (4.3)$$

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \rightarrow E_{ji}^*, \quad (4.4)$$

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \rightarrow E_{ij} E_{kl}, \quad (4.5)$$

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \rightarrow E_{ij} U_s a'_r(s) U_s^*, \quad (4.6)$$

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \rightarrow U_s a'_r(s) U_s^* E_{ij} \quad (4.7)$$

are measurable from  $X \times \mathcal{C}$  to  $\mathcal{B}$  when  $\mathcal{C}$  is equipped with the Borel structure obtained from the product topology. By Lemma 14.3.1 in [16], there is a Borel  $\mu$ -null subset  $X_1$  of  $X$  such that, when restricted to  $X \setminus X_1$ , these maps are all Borel measurable.

Let  $N = X_0 \cup X_1$ . It follows that  $\mu(N) = 0$ . Let  $\eta$  be the set of elements

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn})) \in (X \setminus N) \times \mathcal{C}$$

satisfying

- (i)  $E_{11} = U_s p(s) U_s^*$ ;
- (ii)  $E_{ij} = E_{ji}^*$  and  $E_{ij} E_{kl} = \delta_{jk} E_{il}$  for any  $i, j, k, l \in \{1, 2, \dots, n\}$ ;
- (iii)  $E_{ij} U_s a'_r(s) U_s^* = U_s a'_r(s) U_s^* E_{ij}$  for any  $i, j \in \{1, 2, \dots, n\}$ .

**Claim 4.2.1:** *The set  $\eta$  is analytic.*

Proof of Claim 4.2.1: Since the maps (4.1)-(4.7) are all Borel measurable when restricted to  $X \setminus N$ ,  $\eta$  is a Borel set. By Theorem 14.3.5 in [16],  $\eta$  is an analytic set.

The proof of Claim 4.2.1 is complete.

**Claim 4.2.2 :** *Let  $\pi$  be the projection of  $X \times \mathcal{C}$  onto  $X$ . Then  $\pi(\eta) = X \setminus N$ .*

Proof of Claim 4.2.2: Let  $(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn}))$  be an element in  $\eta$ . Since  $\{a'_r(s) : r \in \mathbb{N}\}$  is strong operator dense in the unit ball  $(\mathcal{M}'_s)_1$  of  $\mathcal{M}'_s$  for every  $s \in X$ , conditions (ii) and (iii) are equivalent to the statement that  $\{U_s^* E_{ij} U_s : 1 \leq i, j \leq n\}$  is a system of matrix units in  $\mathcal{M}_s$ .

Note  $X_0$  is a  $\mu$ -null subset of  $X$  and  $\{p_{i,s} \in \mathcal{M}_s : 2 \leq i \leq n, s \in X\}$  is a family of projections satisfying  $p(s), p_{2,s}, \dots, p_{n,s}$  are  $n$  orthogonal equivalent projections in  $\mathcal{M}_s$  for all  $s \in X \setminus X_0$ . Hence for each  $s \in X \setminus X_0$ , there is a system of matrix units  $\{E_{ij} : 1 \leq i, j \leq n\}$  such that (a)  $E_{11} = p(s)$  and (b)  $E_{ii} = p_{i,s}$  for each  $2 \leq i \leq n$ . From arguments in the preceding paragraph,

it follows that

$$(s, (E_{11}, E_{12}, \dots, E_{1n}, E_{21}, \dots, E_{2n}, \dots, E_{nn}))$$

satisfies conditions (i), (ii) and (iii) for every  $s \in X \setminus N$ . Therefore  $\pi(\eta) = X \setminus N$ .

The proof of Claim 4.2.2 is complete.

(Continue the proof of Lemma 4.2) By Claim 4.2.1 and Claim 4.2.2,  $\eta$  is analytic and  $\pi(\eta) = X \setminus N$ . It follows from Theorem 14.3.6 in [16] that, there is a measurable map

$$s \rightarrow (E_{11,s}, E_{12,s}, \dots, E_{1n,s}, E_{21,s}, \dots, E_{2n,s}, \dots, E_{nn,s})$$

from  $X \setminus N$  to  $\mathcal{C}$  such that

$$(s, (E_{11,s}, E_{12,s}, \dots, E_{1n,s}, E_{21,s}, \dots, E_{2n,s}, \dots, E_{nn,s})) \in \eta$$

for  $s \in X \setminus N$ . Defining  $E_{ij,s} = 0$  for  $s \in N$  and  $i, j \in \{1, 2, \dots, n\}$ , we get a measurable map

$$s \rightarrow (E_{11,s}, E_{12,s}, \dots, E_{1n,s}, E_{21,s}, \dots, E_{2n,s}, \dots, E_{nn,s})$$

from  $X$  to  $\mathcal{C}$  satisfying

$$(s, (E_{11,s}, E_{12,s}, \dots, E_{1n,s}, E_{21,s}, \dots, E_{2n,s}, \dots, E_{nn,s})) \in \eta \quad \text{for almost every } s \in X. \quad (4.8)$$

For all  $s \in X$ , all  $i, j \in \{1, 2, \dots, n\}$  and two vectors  $x, y \in H$ , we have

$$\langle U_s^* E_{ij,s} U_s x(s), y(s) \rangle = \langle E_{ij,s} U_s x(s), U_s y(s) \rangle.$$

Thus from (4.8), it follows that the map  $s \rightarrow \langle U_s^* E_{ij,s} U_s x(s), y(s) \rangle$  is measurable. Since

$$|\langle U_s^* E_{ij,s} U_s x(s), y(s) \rangle| \leq \|x(s)\| \|y(s)\|,$$

the map  $s \rightarrow \langle U_s^* E_{ij,s} U_s x(s), y(s) \rangle$  is integrable. By Definition 14.1.1 in [16],  $U_s^* E_{ij,s} U_s x(s) = (p_{ij}x)(s)$  almost everywhere for some  $p_{ij}x \in H$ . Therefore  $p_{ij}(s) = U_s^* E_{ij,s} U_s$  for almost every  $s \in X$ . It follows from condition (iii) that  $p_{ij} \in \mathcal{M}$ . For any  $i \in \{2, 3, \dots, n\}$ , let  $p_i = p_{ii}$ . From  $p_{ii}(s) = U_s^* E_{ii,s} U_s$ , it follows from (i) and (ii) that  $p_2, p_3, \dots, p_n$  are the required projections.  $\square$

LEMMA 4.3. Let  $\mathcal{M}$  be a type  $II_1$  von Neumann algebra acting on a separable Hilbert space  $H$ . Let  $\mathcal{M} = \int_X \bigoplus \mathcal{M}_s d\mu$  and  $H = \int_X H_s d\mu$  be the direct integral decompositions of  $\mathcal{M}$  and  $H$  as in Lemma 2.4. Suppose that  $\mathcal{M}_s$  is a type  $II_1$  factor with a trace  $\tau_s$  for every  $s \in X$ . Let  $n \in \mathbb{N}$  and  $\epsilon$  be a positive number. Let  $x = (x_{ij})$  be an element in  $M_n(\mathcal{M})$  such that, for every  $s \in X$ ,  $\sum_{i,j=1}^n \|x_{ij}(s)\|_{2,s}^2 < \epsilon^2$ , where the  $\|\cdot\|_{2,s}$  is the 2-norm on  $\mathcal{M}_s$  induced by  $\tau_s$ . Then there are  $n$  equivalent orthogonal projections  $\{p_1, p_2, \dots, p_n\}$  in  $\mathcal{M}$  and  $n$  equivalent orthogonal projections  $q_1, q_2, \dots, q_n$  in  $\mathcal{M}$  such that

$$x_{ij} = p_1 x_{ij} q_1 + h_{ij}$$

with  $\|h_{ij}\| \leq 3\epsilon\sqrt{n}$  for every  $i, j \in \{1, 2, \dots, n\}$ .

PROOF. We will use Pisier's trick in [20] to prove the result.

Take  $a = (\sum_{i,j=1}^n x_{ij}x_{ij}^*)^{1/2}$ . Let  $\mathcal{A}$  be the unital  $C^*$ -subalgebra of  $\mathcal{M}$  generated by  $\{x_{ij} : 1 \leq i, j \leq n\}$ . By Theorem 14.1.13 in [16] and the fact that  $\mathcal{A}$  is a separable  $C^*$ -algebra, we know that  $\mathcal{A} = \int_X \bigoplus \mathcal{A}_s d\mu$  and the map  $y \rightarrow y(s)$  from  $\mathcal{A}$  to  $\mathcal{A}_s$  is a unital  $*$ -homomorphism for almost every  $s \in X$ . It follows that  $a(s) = (\sum_{i,j=1}^n x_{ij}(s)x_{ij}(s)^*)^{1/2}$  for almost every  $s \in X$ .

Without loss of generality, we might assume  $a(s) = (\sum_{i,j=1}^n x_{ij}(s)x_{ij}(s)^*)^{1/2}$  for every  $s \in X$ . Since

$$\sum_{i,j=1}^n \|x_{ij}(s)\|_{2,s}^2 < \epsilon^2, \text{ we know}$$

$$\|a(s)\|_{2,s} < \epsilon, \quad \text{for every } s \in X. \quad (4.9)$$

Note that  $a$  is a positive element in  $\mathcal{M}$ . By functional calculus, we know there exist an positive integer  $k$ , a family of positive numbers  $\lambda_1, \dots, \lambda_k$  and orthogonal projections  $P_1, \dots, P_k$  in  $\mathcal{M}$  such that

$$\|a - \sum_{i=1}^k \lambda_i P_i\| < \epsilon \quad \text{and} \quad \|a^2 - \sum_{i=1}^k \lambda_i^2 P_i\| < \epsilon. \quad (4.10)$$

Thus  $P_1(s), \dots, P_k(s)$  are orthogonal projections in  $\mathcal{M}_s$  and

$$\|a(s) - \sum_{i=1}^k \lambda_i P_i(s)\| < \epsilon, \quad (4.11)$$

for almost every  $s \in X$ .

From (4.9) and (4.11), it follows that

$$\|\sum_{i=1}^k \lambda_i P_i(s)\|_{2,s} < 2\epsilon, \quad \text{for almost every } s \in X. \quad (4.12)$$

Denote

$$p_1 = \chi_{[2\epsilon\sqrt{n}, \infty)}(\sum_{i=1}^k \lambda_i P_i). \quad (4.13)$$

We have

$$p_1(s) = \left( \chi_{[2\epsilon\sqrt{n}, \infty)}(\sum_{i=1}^k \lambda_i P_i) \right)(s) = \chi_{[2\epsilon\sqrt{n}, \infty)}(\sum_{i=1}^k \lambda_i P_i(s)), \quad \text{for almost every } s \in X. \quad (4.14)$$

By (4.12) and (4.14),

$$\tau_s(p_1(s)) < 1/n, \quad \text{for almost every } s \in X.$$

Since  $\mathcal{M}_s$  is a type  $II_1$  factor for every  $s \in X$ , there exist projections  $p_{2,s}, p_{3,s}, \dots, p_{n,s}$  in  $\mathcal{M}_s$  such that  $\{p_1(s), p_{2,s}, \dots, p_{n,s}\}$  are orthogonal equivalent projections in  $\mathcal{M}_s$ . By Lemma 4.2,

there exist projections  $p_2, p_3, \dots, p_n$  in  $\mathcal{M}$  such that  $p_1, p_2, \dots, p_n$  are  $n$  orthogonal equivalent projections in  $\mathcal{M}$ .

Take  $b = (\sum_{i,j=1}^n x_{ij}^* x_{ij})^{1/2}$ . Similarly, we assume that  $b(s) = (\sum_{i,j=1}^n x_{ij}(s)^* x_{ij}(s))^{1/2}$  for every  $s \in X$ .

Again, note that  $b$  is a positive element in  $\mathcal{M}$ . By functional calculus, we know there exist an positive integer  $k'$ , a family of positive numbers  $\alpha_1, \dots, \alpha_{k'}$  and orthogonal projections  $Q_1, \dots, Q_{k'}$  in  $\mathcal{M}$  such that

$$\|b - \sum_{i=1}^{k'} \alpha_i Q_i\| < \epsilon \quad \text{and} \quad \|b^2 - \sum_{i=1}^{k'} \alpha_i^2 Q_i\| < \epsilon. \quad (4.15)$$

Without loss of generality, we can further assume that

$$\|b(s) - \sum_{i=1}^{k'} \alpha_i Q_i(s)\| < \epsilon, \quad \forall s \in X. \quad (4.16)$$

By a similar argument as the last paragraph, we obtain a spectral projection

$$q_1 = \chi_{[2\epsilon\sqrt{n}, \infty)}(\sum_{i=1}^{k'} \alpha_i Q_i)$$

and projections  $q_2, q_3, \dots, q_n$  such that  $q_1, q_2, \dots, q_n$  are  $n$  orthogonal equivalent projections in  $\mathcal{M}$  and  $q_1(s) = \chi_{[2\epsilon\sqrt{n}, \infty)}(\sum_{i=1}^{k'} \alpha_i Q_i(s))$  for almost  $s \in X$ .

Take  $h_{ij} = x_{ij} - p_1 x_{ij} q_1$ . We may assume that  $h_{ij}(s) = x_{ij}(s) - p_1(s) x_{ij}(s) q_1(s)$  for every  $s \in X$ . In the following we show that  $\|h_{ij}\| \leq 3\epsilon\sqrt{n}$ . By Proposition 14.1.9 in [16],  $\|h_{ij}\|$  is the essential bound of  $\{\|h_{ij}(s)\| : s \in X\}$ . So it suffices to show that  $\|h_{ij}(s)\| \leq 3\epsilon\sqrt{n}$  for almost every  $s \in X$ . For every  $s \in X$ ,

$$\|h_{ij}(s)\| = \|x_{ij}(s) - p_1(s) x_{ij}(s) q_1(s)\| \leq \|(1 - p_1(s)) x_{ij}(s)\| + \|p_1(s) x_{ij}(s) (1 - q_1(s))\|. \quad (4.17)$$

We have, from (4.10) and (4.15),

$$\begin{aligned} & \|(1 - p_1(s)) x_{ij}(s)\|^2 \\ &= \|(1 - p_1(s)) x_{ij}(s) x_{ij}(s)^* (1 - p_1(s))\|^2 \\ &\leq \|(1 - p_1(s)) a(s)^2 (1 - p_1(s))\|^2 \\ &\leq \|(1 - p_1(s)) \left( \sum_{i=1}^k \lambda_i^2 P_i + (a(s)^2 - \sum_{i=1}^k \lambda_i^2 P_i) \right) (1 - p_1(s))\|^2 \\ &\leq 4\epsilon^2 n + \epsilon^2 \leq 5\epsilon^2 n \end{aligned} \quad (4.18)$$

and

$$\begin{aligned}
& \|p_1(s)x_{ij}(s)(1 - q_1(s))\|^2 \\
& \leq \|x_{ij}(s)(1 - q_1(s))\|^2 \\
& = \|(1 - q_1(s))x_{ij}(s)^*x_{ij}(s)(1 - q_1(s))\| \\
& \leq 5\epsilon^2 n,
\end{aligned} \tag{4.19}$$

for almost every  $s \in X$ .

It follows from (4.17), (4.18) and (4.19) that for any  $i, j \in \{1, 2, \dots, n\}$ ,

$$\|h_{ij}\| \leq 3\epsilon\sqrt{n}.$$

The proof is complete.  $\square$

Now we are ready to prove the following result as a generalization of Theorem 13 in [20].

**THEOREM 4.4.** *Let  $\mathcal{M}$  be a countably decomposable type  $II_1$  von Neumann algebra. If  $\mathcal{M}$  has Property  $\Gamma$ , then  $d(\mathcal{M}) \leq 5$ .*

**PROOF.** To show that  $d(\mathcal{M}) = l(\mathcal{M}) \leq 5$ , it suffices to prove that there exists a positive scalar  $k$  such that for any  $n \in \mathbb{N}$  and any  $x = (x_{ij}) \in M_n(\mathcal{M})$ ,

$$\|x\|_{(5, \mathcal{M})} \leq k\|x\|.$$

In the following we fix  $n \in \mathbb{N}$  and  $x = (x_{ij}) \in M_n(\mathcal{M})$ . We may assume that  $\|x\| = 1$ .

Let  $F = \{x_{ij}\}_{i,j=1}^n$  be a finite subset of  $\mathcal{M}$ . Note  $\mathcal{M}$  is a countably decomposable type  $II_1$  von Neumann algebra with Property  $\Gamma$ . From Lemma 3.6, it follows that there exists a von Neumann algebra  $\mathcal{M}_1$  with separable predual and with Property  $\Gamma$  such that  $F \subseteq \mathcal{M}_1 \subseteq \mathcal{M}$ . It is easy to see from Definition 4.1 that

$$\|x\|_{(5, \mathcal{M})} \leq \|x\|_{(5, \mathcal{M}_1)}.$$

Hence, to prove the theorem, it suffices to show that there exists a universal constant  $k > 0$  such that

$$\|x\|_{(5, \mathcal{M}_1)} \leq k.$$

Replacing  $\mathcal{M}$  by  $\mathcal{M}_1$ , we can assume that  $\mathcal{M}$  has separable predual and Property  $\Gamma$ .

Since  $\mathcal{M}$  has separable predual, by Proposition A.2.1 in [14], there is a faithful normal representation  $\pi$  of  $\mathcal{M}$  on a separable Hilbert space  $H$ . Replacing  $\mathcal{M}$  by  $\pi(\mathcal{M})$  if necessary, we assume that  $\mathcal{M}$  is acting on a separable Hilbert space.

Let  $\mathcal{Z}_{\mathcal{M}}$  be the center of  $\mathcal{M}$ . Let  $\mathcal{M} = \int_X \bigoplus \mathcal{M}_s d\mu$  and  $H = \int_X \bigoplus H_s d\mu$  be the direct integral decompositions of  $\mathcal{M}$  and  $H$  relative to  $\mathcal{Z}_{\mathcal{M}}$  as in Lemma 2.4. By Proposition 3.12 in [23], we may assume that  $\mathcal{M}_s$  is a type  $II_1$  factor with Property  $\Gamma$  for every  $s \in X$ . For any  $\epsilon > 0$ , applying Corollary 4.2 in [23], we obtain  $n$  orthogonal equivalent projections  $p_1, p_2, \dots, p_n$  in  $\mathcal{M}$  summing to  $I$  such that

$$\sum_{i,j=1}^n \|(x_{ij} - \sum_{m=1}^n p_m x_{ij} p_m)(s)\|_{2,s}^2 < \epsilon^2, \tag{4.20}$$



where  $\|\cdot\|_{2,s}$  is the 2-norm induced by the unique trace  $\tau_s$  on  $\mathcal{M}_s$  for  $s \in X$  almost everywhere. Decompose

$$x = \left(\sum_{m=1}^n p_m x_{ij} p_m\right) + \left(x_{ij} - \sum_{m=1}^n p_m x_{ij} p_m\right). \quad (4.21)$$

Let  $x_m = (p_m x_{ij} p_m)$  for  $1 \leq m \leq n$ . Therefore  $x_m = (1 \otimes p_m)x(1 \otimes p_m)$  for  $1 \leq m \leq n$ . By applying Lemma 5 in [20], for each  $m$ ,

$$\|x_m\|_{(3,\mathcal{M})} \leq 1.$$

Since  $\left(\sum_{m=1}^n p_m x_{ij} p_m\right) = \left(\sum_{m=1}^n p_m (p_m x_{ij} p_m) p_m\right)$ , by Lemma 14 in [20],

$$\left\|\left(\sum_{m=1}^n p_m x_{ij} p_m\right)\right\|_{(5,\mathcal{M})} \leq 1. \quad (4.22)$$

Let  $x'_{ij} = x_{ij} - \sum_{m=1}^n p_m x_{ij} p_m$  for each  $i, j \in \{1, 2, \dots, n\}$ . It follows that  $\|(x'_{ij})\| \leq 2$ . By Lemma 4.2, (4.20) implies that there exist  $n$  orthogonal equivalent projections  $p'_1, p'_2, \dots, p'_n$  and  $n$  orthogonal equivalent projections  $q'_1, q'_2, \dots, q'_n$  such that

$$x'_{ij} = p'_1 x'_{ij} q'_1 + h'_{ij}$$

with  $\|h'_{ij}\| \leq 3\epsilon\sqrt{n}$ . By Lemma 5 in [20],

$$\|(p'_1 x'_{ij} q'_1)\|_{(5,\mathcal{M})} \leq \|(p'_1 x'_{ij} q'_1)\|_{(3,\mathcal{M})} = \|(1 \otimes p'_1)(x'_{ij})(1 \otimes q'_1)\|_{(3,\mathcal{M})} \leq 2. \quad (4.23)$$

Since  $\|h'_{ij}\| \leq 3\epsilon\sqrt{n}$  for every  $i, j \in \{1, 2, \dots, n\}$ , we obtain

$$\begin{aligned} \|(h'_{ij})\|_{(5,\mathcal{M})} &\leq n\|(h'_{ij})\| \\ &\leq n^3 \sup\{\|h'_{ij}\| : i, j \in \{1, 2, \dots, n\}\} \\ &\leq 3n^3\sqrt{n}\epsilon. \end{aligned} \quad (4.24)$$

By (4.21), (4.22), (4.23) and (4.24),  $\|x\|_{(5,\mathcal{M})} \leq 3 + 3n^3\sqrt{n}\epsilon$ . Since  $\epsilon$  was arbitrarily chosen, we obtain that  $\|x\|_{(5,\mathcal{M})} \leq 3$ . Therefore

$$d(\mathcal{M}) \leq 5.$$

The proof is complete. □

## 5. Similarity degree of a class of C\*-algebras

In this section we will deal with the similarity degree of a class of C\*-algebras with certain properties. We will conclude that  $d(\mathcal{A}) \leq 3$  if  $\mathcal{A}$  is a separable unital  $\mathcal{Z}$ -stable C\*-algebra.

Before we prove the main theorem of this section, we give the following lemma.

LEMMA 5.1. *Let  $\mathcal{M}$  be a von Neumann algebra with the type decomposition*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_{c_1} \oplus \mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty,$$

*where  $\mathcal{M}_1$  is a type I von Neumann algebra,  $\mathcal{M}_{c_1}$  is a type  $II_1$  von Neumann algebra,  $\mathcal{M}_{c_\infty}$  is a type  $II_\infty$  von Neumann algebra and  $\mathcal{M}_\infty$  is a type III von Neumann algebra. Suppose  $\mathcal{M}_{c_1}$  has finite similarity degree. If  $\phi$  is a bounded unital representation of  $\mathcal{M}$  on a Hilbert space  $H$ , which is continuous from  $\mathcal{M}$ , with the topology  $\sigma(\mathcal{M}, \mathcal{M}_\#)$ , to  $B(H)$ , with the topology  $\sigma(B(H), B(H)_\#)$ , then  $\phi$  is completely bounded.*

PROOF. Observe that there is no trace on  $\mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty$ . It follows from Theorem 8 in [20] that  $\mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty$  has finite similarity degree. From the hypothesis,  $\mathcal{M}_{c_1}$  has finite similarity degree.

Since  $\mathcal{M}_1$  is of type I, there is a directed collection  $\{\mathcal{A}_i : i \in I\}$  of finite dimensional unital  $C^*$ -subalgebras such that  $\cup_{i \in I} \mathcal{A}_i$  is dense in  $\mathcal{M}_1$  under the topology  $\sigma(\mathcal{M}, \mathcal{M}_\#)$ . Let  $\mathcal{A}$  be the norm closure of  $\cup_{i \in I} \mathcal{A}_i$ . Then  $\mathcal{A}$  is also dense in  $\mathcal{M}_1$  under the topology  $\sigma(\mathcal{M}, \mathcal{M}_\#)$ . Each  $\mathcal{A}_i$  is finite dimensional, therefore  $\mathcal{A}_i$  is nuclear for each  $i \in I$ . It follows from Proposition 11.3.12 in [16] that  $\mathcal{A}$  is nuclear. By Theorem 4.1 in [3],  $\mathcal{A}$  has finite similarity degree.

Now let  $\mathcal{B} = \mathcal{A} \oplus \mathcal{M}_{c_1} \oplus \mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty$ . Then  $\mathcal{B}$  is dense in  $\mathcal{M}$  under the topology  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  and we can obtain that  $\mathcal{B}$  has finite similarity degree by Remark 6 in [19]. Thus the restriction of  $\phi$  to  $\mathcal{B}$  is completely bounded. Notice that  $\phi$  is continuous from  $\mathcal{M}$ , with the topology  $\sigma(\mathcal{M}, \mathcal{M}_\#)$ , to  $B(H)$ , with the topology  $\sigma(B(H), B(H)_\#)$ . Now we conclude that  $\phi$  is a completely bounded representation of  $\mathcal{M}$  on  $H$ . □

The following Theorem gives an estimation of  $\|\phi\|_{cb}$  in the case that  $\mathcal{M}_{c_1}$  is a countably decomposable type  $II_1$  von Neumann algebra with Property  $\Gamma$ . The main idea of the proof of the following theorem is based on the one used by Christensen in [6].

THEOREM 5.2. *Let  $\mathcal{M}$  be a von Neumann algebra with the type decomposition*

$$\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_{c_1} \oplus \mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty,$$

*where  $\mathcal{M}_1$  is a type I von Neumann algebra,  $\mathcal{M}_{c_1}$  is a type  $II_1$  von Neumann algebra,  $\mathcal{M}_{c_\infty}$  is a type  $II_\infty$  von Neumann algebra and  $\mathcal{M}_\infty$  is a type III von Neumann algebra. Suppose  $\mathcal{M}_{c_1}$  is a countably decomposable von Neumann algebra with Property  $\Gamma$ . If  $\phi$  is a bounded unital representation of  $\mathcal{M}$  on a Hilbert space  $H$ , which is continuous from  $\mathcal{M}$ , with the topology  $\sigma(\mathcal{M}, \mathcal{M}_\#)$ , to  $B(H)$ , with the topology  $\sigma(B(H), B(H)_\#)$ , then  $\phi$  is completely bounded and  $\|\phi\|_{cb} \leq \|\phi\|^3$ .*

PROOF. By Theorem 4.4 we can obtain that  $\mathcal{M}_{c_1}$  has finite similarity degree. It follows directly from Lemma 5.1 that  $\phi$  is completely bounded. Therefore there exists an invertible positive operator  $t \in B(H)$  such that  $\pi(\cdot) = t\phi(\cdot)t^{-1}$  is a unital  $*$ -homomorphism. Since  $\phi$  is continuous from  $\mathcal{M}$  in  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  topology to  $B(H)$  in  $\sigma(B(H), B(H)_\#)$  topology,  $\pi$  is continuous from  $\mathcal{M}$  in  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  topology to  $B(H)$  in  $\sigma(B(H), B(H)_\#)$  topology.

Let  $\mathcal{I} = \ker(\pi)$ . Then  $\mathcal{I}$  is a two-sided ideal in  $\mathcal{M}$  and is closed in  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  topology. From Proposition 1.10.5 in [25], it follows that there is a central projection  $p$  in  $\mathcal{M}$  such that

$\mathcal{I} = p\mathcal{M}$ . Let  $\phi_p, \pi_p : (I - p)\mathcal{M} \rightarrow B(H)$  be defined by

$$\phi_p(a) = \phi(a) \quad \text{and} \quad \pi_p(a) = \pi(a) \quad \text{for } a \in (I - p)\mathcal{M}.$$

By the choice of  $p$ , we know that  $\phi_p$  and  $\pi_p$  are injective homomorphisms from  $(I - p)\mathcal{M}$  to  $B(H)$  such that  $\|\phi_p\| = \|\phi\|$ ,  $\|\phi_p\|_{cb} = \|\phi\|_{cb}$ , and  $\phi_p(\cdot) = t\pi_p(\cdot)t^{-1}$ . Moreover, by the fact that  $p$  is a central projection, we have

$$(I - p)\mathcal{M} = (I - p)\mathcal{M}_1 \oplus (I - p)\mathcal{M}_{c_1} \oplus (I - p)\mathcal{M}_{c_\infty} \oplus (I - p)\mathcal{M}_\infty,$$

and  $(I - p)\mathcal{M}_{c_1}$  is a type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ . Replacing  $\mathcal{M}, \phi$ , and  $\pi$  by  $(I - p)\mathcal{M}, \phi_p$ , and  $\pi_p$  if necessary, we might assume that

$\phi$  is an injective homomorphism and  $\pi$  is a  $*$ -isomorphism.

Dividing  $t$  by  $\|t\|$  if necessary, we assume that  $\|t\| = 1$ .

Let  $(\mathcal{M})_1$  be the unit ball of  $\mathcal{M}$  and  $(\pi(\mathcal{M}))_1$  be the unit ball of  $\pi(\mathcal{M})$ . Note  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  topology (or  $\sigma(B(H), B(H)_\#)$  topology) coincides with the weak operator topology (*WOT*) on bounded subsets of  $\mathcal{M}$  (or  $B(H)$  respectively). Since  $\pi$  is continuous from  $\mathcal{M}$  in  $\sigma(\mathcal{M}, \mathcal{M}_\#)$  topology to  $B(H)$  in  $\sigma(B(H), B(H)_\#)$  topology,  $\pi$  is *WOT-WOT* continuous when restricted to bounded subsets of  $\mathcal{M}$ . Therefore  $\pi(\mathcal{M})$  is a von Neumann algebra.

Let  $p_1, p_2, p_3$  be central projections in  $\mathcal{M}$  with sum  $I$  such that  $p_1\mathcal{M} = \mathcal{M}_1$ ,  $p_2\mathcal{M} = \mathcal{M}_{c_1}$  and  $p_3\mathcal{M} = \mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty$ . By assumption on  $\mathcal{M}$ , we know that  $\mathcal{M}_{c_1}$  is a type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ . Moreover

$$q_i = \pi(p_i)$$

is a central projection in  $\pi(\mathcal{M})$  for each  $i = 1, 2, 3$  and we can decompose

$$\pi = \pi_1 \oplus \pi_2 \oplus \pi_3,$$

where

$$\pi_i(a) = \pi(p_i a) = q_i \pi(a), \quad \forall a \in \mathcal{M}, \quad i = 1, 2, 3.$$

Since  $\pi$  is *WOT-WOT* continuous when restricted to bounded subsets of  $\mathcal{M}$ , each  $\pi_i$  is *WOT-WOT* continuous on bounded sets of  $\mathcal{M}$ . If one of  $\pi_1, \pi_2, \pi_3$  is trivial, the following proof will be simplified. Here we assume that they are all nontrivial. By a similar argument as the preceding paragraph, we obtain that  $\pi_i(\mathcal{M})$  is a von Neumann algebra over  $q_i H$  for  $i = 1, 2, 3$ . Since  $\pi$  is a  $*$ -isomorphism from  $\mathcal{M}_{c_1}$  onto  $\pi(\mathcal{M}_{c_1}) = \pi_2(\mathcal{M}_{c_1})$  and  $\mathcal{M}_{c_1}$  is a countably decomposable type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ ,  $\pi_2(\mathcal{M}_{c_1})$  is a countably decomposable type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ .

Let  $c = \|\phi\|$ . It follows that for any  $a \in (\mathcal{M})_1$ ,

$$\phi(a)\phi(a)^* \leq c^2.$$

Since  $\pi(\cdot) = t\phi(\cdot)t^{-1}$  and  $t$  is positive, we obtain that for any  $a \in (\mathcal{M})_1$ ,

$$\pi(a)t^2\pi(a)^* \leq c^2t^2. \tag{5.1}$$

Since  $\pi$  is a  $*$ -isomorphism between  $\mathcal{M}$  and  $\pi(\mathcal{M})$ , we know that  $\pi((\mathcal{M})_1) = (\pi(\mathcal{M}))_1$ . Now, equation (5.1) implies that for any unitary  $u \in \pi(\mathcal{M})$ ,  $ut^2u^* \leq c^2t^2$ . Therefore for any unitaries  $u, v \in \pi(\mathcal{M})$ ,

$$ut^2u^* \leq c^2vt^2v^*. \quad (5.2)$$

To show  $\|\phi\|_{cb} \leq c^3$ , we just need to prove that for any  $n \in \mathbb{N}$  and any unitary  $U \in M_n(\mathcal{M})$ ,

$$\pi^{(n)}(U)(I_n \otimes t^2)\pi^{(n)}(U^*) \leq c^6(I_n \otimes t^2), \quad (5.3)$$

where  $\pi^{(n)} : M_n(\mathcal{M}) \rightarrow B(H^n)$  is the  $n$ -folded map of  $\pi$ .

Now fix  $n \in \mathbb{N}$ ,  $\epsilon > 0$  and a unitary  $U \in M_n(\mathcal{M})$ . Let  $x = (x_1, x_2, \dots, x_n)$  be a unit vector in  $H^n$ . Denote

$$x^{(i)} = (I_n \otimes p_i)x = (x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)}), \quad i = 1, 2, 3.$$

In the following we will carry out computations according to the representations  $\pi_1, \pi_2, \pi_3$  and prove inequality (5.3) in each case.

We will first replace  $t^2$  by an element that commutes with  $q_1, q_2$  and  $q_3$ .

Note  $\pi_2(\mathcal{M}_{c_1})$  is a countably decomposable type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ . Thus there exists a faithful normal tracial state  $\rho$  on  $\pi_2(\mathcal{M}_{c_1})$ . Let  $\|\cdot\|_2$  be the 2-norm induced by  $\rho$  on  $\pi_2(\mathcal{M}_{c_1})$ . Then 2-norm topology coincides with the strong operator topology on bounded subsets of  $\pi_2(\mathcal{M}_{c_1})$ . It follows from Corollary 3.4 in [23] that we can obtain  $n$  orthogonal equivalent projections  $r_1, r_2, \dots, r_n$  in  $\pi_2(\mathcal{M}_{c_1})$  with sum  $q_2$  such that, for each  $i \in \{1, 2, \dots, n\}$ ,

$$\|(I_n \otimes (1 - r_i))(\pi_2)^{(n)}(U_2)(I_n \otimes r_i)x^{(2)}\| \leq (\frac{\epsilon}{n^3})^{1/2}. \quad (5.4)$$

From the fact that projections  $r_1, r_2, \dots, r_n$  are orthogonal equivalent projections in  $\pi_2(\mathcal{M}_{c_1})$ , there exists a system of matrix units  $\{r_{ij} : i, j = 1, 2, \dots, n\} \subseteq \pi_2(\mathcal{M}_{c_1})$  in such that  $r_{ii} = r_i$  for each  $i$ . Denote by  $\mathcal{N}$  the von Neumann algebra generated by  $\{r_{ij} : i, j = 1, 2, \dots, n\}$ . It follows that  $\mathcal{N} \cong M_n(\mathbb{C})$  and  $q_2 \in \mathcal{N}$ .

Since  $\mathcal{M}_{c_\infty}$  is of type II <sub>$\infty$</sub>  and  $\mathcal{M}_\infty$  is of type III, there exist subfactors  $\mathcal{B}_2$  in  $\mathcal{M}_{c_\infty}$  and  $\mathcal{B}_3$  in  $\mathcal{M}_\infty$  such that  $\mathcal{B}_2 \cong \mathcal{B}_3 \cong B(l^2(\mathbb{N}))$ . Hence, there exists a subfactor  $\mathcal{B}$  of  $\pi_3(\mathcal{M}) (= \pi_3(\mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty))$  such that  $\mathcal{B} \cong B(l^2(\mathbb{N}))$  and  $q_3 \in \mathcal{B}$ . Take  $\mathcal{K} = \mathcal{B}' \cap \pi_3(\mathcal{M})$ . By Lemma 11.4.11 in [16],

$$\pi_3(\mathcal{M}) \cong \mathcal{B} \otimes \mathcal{K}.$$

Let  $\mathcal{R} = \pi_1(\mathcal{M}) \oplus \mathcal{N} \oplus \mathcal{B}$ . Since  $\mathcal{M}_1$  is of type I, the von Neumann algebra  $\pi_1(\mathcal{M}) (= \pi_1(\mathcal{M}_1))$  is of type I. By the choices of  $\mathcal{B}$  and  $\mathcal{N}$ , we know that  $\mathcal{R}$  is a type I von Neumann algebra and thus an injective von Neumann algebra. Let  $\mathcal{U}$  be the unitary group of  $\mathcal{R}$ . Hence the set

$$W = \overline{\text{conv}}^{uw} \{ut^2u^* : u \in \mathcal{U}\} \cap \mathcal{R}' \quad (5.5)$$

is nonempty, where  $\overline{\text{conv}}^{uw} \{ut^2u^* : u \in \mathcal{U}\}$  denotes the closure of the convex hull of the set  $\{ut^2u^* : u \in \mathcal{U}\}$  in the ultraweak topology. Choose  $m \in W$ . It is clear to see that  $\|m\| \leq \|t^2\| \leq 1$ . By equation (5.2), for any unitaries  $u, v \in \pi(\mathcal{M})$ ,

$$umu^* \leq c^2vmv^*. \quad (5.6)$$

Since  $q_1, q_2$  and  $q_3$  are in  $\mathcal{R}$ , we obtain that  $m$  commutes with  $q_1, q_2$  and  $q_3$ . Let

$$m_1 = q_1 m, \quad m_2 = q_2 m \quad \text{and} \quad m_3 = q_3 m.$$

Therefore inequality (5.6) implies

(i) for any unitaries  $u, v \in \pi_1(\mathcal{M}_1)$ ,

$$um_1u^* \leq c^2vm_1v^*; \quad (5.7)$$

(ii) for any unitaries  $u, v \in \pi_2(\mathcal{M}_{c_1})$ ,

$$um_2u^* \leq c^2vm_2v^*; \quad (5.8)$$

(iii) for any unitaries  $u, v \in \pi_3(\mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty)$ ,

$$um_3u^* \leq c^2vm_3v^*. \quad (5.9)$$

Since  $m \in \mathcal{R}'$ , it commutes with  $\pi_1(\mathcal{M})$ . Thus  $I_n \otimes m_1$  commutes with  $\pi_1^{(n)}(\mathcal{M})$ . So we obtain that

$$(\pi_1)^{(n)}(U)(I_n \otimes m_1)(\pi_1)^{(n)}(U^*) = (I_n \otimes m_1)(\pi_1)^{(n)}(U)(\pi_1)^{(n)}(U^*) = I_n \otimes m_1. \quad (5.10)$$

Since  $m \in \mathcal{R}'$  and  $\mathcal{K} = \mathcal{B}' \cap \pi_3(\mathcal{M})$ , we get  $m_3 \in \mathcal{K}$ . Combining inequality (5.9) with the facts that  $\mathcal{B} \cong B(l^2(\mathbb{N}))$ ,  $\pi_3(\mathcal{M}) \cong \mathcal{B} \otimes \mathcal{K}$  and  $m \in \mathcal{K}$ , we conclude that

$$(\pi_3)^{(n)}(U)(I_n \otimes m_3)(\pi_3)^{(n)}(U^*) \leq c^2(I_n \otimes m_3). \quad (5.11)$$

In the following we will deal with  $(\pi_2)^{(n)}$  and  $\mathcal{M}_{c_1}$ . We will use arguments similar to the ones used in the proof of Lemma 5 in [20] and in the proof of Theorem 2.3 in [6].

Recall  $r_1, r_2, \dots, r_n$  are orthogonal equivalent projections in  $\pi_2(\mathcal{M}_{c_1})$  such that

$$r_1 + r_2 + \dots + r_n = q_2. \quad (5.12)$$

Also recall that  $\{r_{ij} : i, j = 1, 2, \dots, n\}$  is a system of matrix units of  $\mathcal{N}$  satisfying, for each  $i \in \{1, 2, \dots, n\}$ , we have  $r_{ii} = r_i$  and

$$\|(I_n \otimes (1 - r_i))(\pi_2)^{(n)}(U_2)(I_n \otimes r_i)x^{(2)}\| \leq \left(\frac{\epsilon}{n^3}\right)^{1/2}. \quad (5.13)$$

Let  $(e_{ij})$  be a system of matrix units for  $M_n(\mathbb{C})$ . For  $k = 1, 2, \dots, n$ , let

$$f_k = \sum_{i=1}^n e_{i1} \otimes r_{ki}.$$

Therefore for each  $k = 1, \dots, n$ ,

$$f_k f_k^* = I_n \otimes r_k, \quad f_k^* f_k = e_{11} \otimes q_2 \quad \text{and} \quad f_k(I_n \otimes m_2) = (I_n \otimes m_2)f_k. \quad (5.14)$$

Since  $\|m\| \leq 1$ , by equation (5.12) and inequalities (5.13), (5.14) we obtain

$$\begin{aligned}
& \langle (\pi_2)^{(n)}(U)(I_n \otimes m_2)(\pi_2)^{(n)}(U^*)x^{(2)}, x^{(2)} \rangle \\
&= \sum_{i,j,k=1}^n \langle (I_n \otimes r_i)(\pi_2)^{(n)}(U)(I_n \otimes r_j)(I_n \otimes m_2)(\pi_2)^{(n)}(U^*)(I_n \otimes r_k)x^{(2)}, x^{(2)} \rangle \\
&\leq \sum_{k=1}^n \langle (I_n \otimes r_k)(\pi_2)^{(n)}(U)(I_n \otimes r_k)(I_n \otimes m_2)(\pi_2)^{(n)}(U^*)(I_n \otimes r_k)x^{(2)}, x^{(2)} \rangle + \epsilon \\
&= \sum_{k=1}^n \langle f_k f_k^* (\pi_2)^{(n)}(U) f_k f_k^* (I_n \otimes m_2) (\pi_2)^{(n)}(U^*) f_k f_k^* x^{(2)}, x^{(2)} \rangle + \epsilon \\
&= \sum_{k=1}^n \langle f_k^* (\pi_2)^{(n)}(U) f_k (I_n \otimes m_2) f_k^* (\pi_2)^{(n)}(U^*) f_k f_k^* x^{(2)}, f_k^* x^{(2)} \rangle + \epsilon.
\end{aligned} \tag{5.15}$$

For each  $k \in \{1, 2, \dots, n\}$ , let

$$y_k = f_k^* x^{(2)} \quad \text{and} \quad a_k = f_k^* (\pi_2)^{(n)}(U) f_k \tag{5.16}$$

By the choice of  $f_k$ , we have

$$a_k = (e_{11} \otimes q_2) a_k (e_{11} \otimes q_2) \quad \text{and} \quad \|a_k\| = \|f_k^* (\pi_2)^{(n)}(U) f_k\| \leq \|f_k^*\| \|f_k\| = 1.$$

Denote  $a_k = e_{11} \otimes a'_k$ , where  $a'_k \in \pi_2(\mathcal{M}_{c_1})$  is a contraction. As  $\pi_2(\mathcal{M}_{c_1})$  is of type  $\text{II}_1$ , we can find two unitaries  $u_k$  and  $v_k$  in  $\pi_2(\mathcal{M}_{c_1})$  such that  $a'_k = \frac{u_k + v_k}{2}$ . So

$$a_k = \frac{1}{2}(e_{11} \otimes (u_k + v_k)), \quad \forall k = 1, \dots, n. \tag{5.17}$$

By inequality (5.8),

$$(e_{11} \otimes u_k)(I_n \otimes m_2)(e_{11} \otimes u_k)^* \leq c^2(e_{11} \otimes m_2) \tag{5.18}$$

and

$$(e_{11} \otimes v_k)(I_n \otimes m_2)(e_{11} \otimes v_k)^* \leq c^2(e_{11} \otimes m_2) \tag{5.19}$$

It follows from (5.15), (5.16) and (5.17) that

$$\begin{aligned}
& \langle (\pi_2)^{(n)}(U)(I_n \otimes m_2)(\pi_2)^{(n)}(U^*)x^{(2)}, x^{(2)} \rangle \\
& \leq \sum_{k=1}^n \langle f_k^*(\pi_2)^{(n)}(U)f_k(I_n \otimes m_2)f_k^*(\pi_2)^{(n)}(U^*)f_k f_k^* x^{(2)}, f_k^* x^{(2)} \rangle + \epsilon \\
& = \frac{1}{4} \sum_{k=1}^n \langle (e_{11} \otimes (u_k + v_k))(I_n \otimes m_2)(e_{11} \otimes (u_k + v_k))^* y_k, y_k \rangle + \epsilon \\
& \leq \frac{1}{4} \sum_{k=1}^n \langle (e_{11} \otimes (u_k + v_k))(I_n \otimes m_2)(e_{11} \otimes (u_k + v_k))^* y_k, y_k \rangle \\
& \quad + \frac{1}{4} \sum_{k=1}^n \langle (e_{11} \otimes (u_k - v_k))(I_n \otimes m_2)(e_{11} \otimes (u_k - v_k))^* y_k, y_k \rangle + \epsilon \\
& = \frac{1}{2} \sum_{k=1}^n \langle (e_{11} \otimes u_k)(I_n \otimes m_2)(e_{11} \otimes u_k)^* y_k, y_k \rangle \\
& \quad + \frac{1}{2} \sum_{k=1}^n \langle (e_{11} \otimes v_k)(I_n \otimes m_2)(e_{11} \otimes v_k)^* y_k, y_k \rangle + \epsilon.
\end{aligned}$$

Thus by (5.18) and (5.19),

$$\begin{aligned}
& \langle (\pi_2)^{(n)}(U)(I_n \otimes m_2)(\pi_2)^{(n)}(U^*)x^{(2)}, x^{(2)} \rangle \\
& \leq c^2 \sum_{k=1}^n \langle (e_{11} \otimes m_2)y_k, y_k \rangle + \epsilon \\
& = c^2 \sum_{k=1}^n \langle f_k(I_n \otimes m_2)f_k^* x^{(2)}, x^{(2)} \rangle + \epsilon.
\end{aligned} \tag{5.20}$$

Since  $f_k$  commutes with  $I_n \otimes m_2$ , we obtain from (5.20) that

$$\begin{aligned}
\langle (\pi_2)^{(n)}(U)(I_n \otimes m_2)(\pi_2)^{(n)}(U^*)x^{(2)}, x^{(2)} \rangle & = c^2 \sum_{k=1}^n \langle f_k f_k^*(I_n \otimes m_2)x^{(2)}, x^{(2)} \rangle + \epsilon \\
& = c^2 \langle (I_n \otimes m_2)x^{(2)}, x^{(2)} \rangle + \epsilon.
\end{aligned} \tag{5.21}$$

Since  $x$  and  $\epsilon$  were arbitrarily chosen, (5.21) implies that

$$(\pi_2)^{(n)}(U)(I_n \otimes m_2)(\pi_2)^{(n)}(U^*) \leq c^2(I_n \otimes m_2). \tag{5.22}$$

Hence by inequalities (5.10), (5.11) and (5.22), we obtain

$$\pi^{(n)}(U)(I_n \otimes m)\pi^{(n)}(U^*) \leq c^2(I_n \otimes m), \quad \text{for all unitary } U \in M_n(\mathcal{M}). \tag{5.23}$$

Since  $m \in W$ , it follows from (5.2) and (5.5) that  $t^2 \leq c^2 m$  and  $m \leq c^2 t^2$ . Therefore (5.23) implies

$$\pi^{(n)}(U)(I_n \otimes t^2)\pi^{(n)}(U^*) \leq c^6(I_n \otimes t^2), \quad \text{for all unitary } U \in M_n(\mathcal{M}).$$

It follows that

$$\|\phi\|_n \leq \|\phi\|^3 \quad \text{for all } n \in \mathbb{N},$$

and

$$\|\phi\|_{cb} \leq \|\phi\|^3.$$

The proof is complete.  $\square$

The class of  $C^*$ -algebras in the following theorem was considered in [11] and it was shown in [11] such  $C^*$ -algebras have similarity degree no more than 11. The following theorem gives a better estimation.

**THEOREM 5.3.** *Suppose  $\mathcal{A}$  is a separable unital  $C^*$ -algebra satisfying*

*Condition (G): if  $\pi$  is a unital  $*$ -representation of  $\mathcal{A}$  on a Hilbert space  $H$  such that  $\pi(\mathcal{A})''$  is a type  $II_1$  factor, then  $\pi(\mathcal{A})''$  has Property  $\Gamma$ , where  $\pi(\mathcal{A})''$  is the von Neumann algebra generated by  $\pi(\mathcal{A})$  in  $B(H)$ .*

*Then  $d(\mathcal{A}) \leq 3$ . Moreover, if  $\mathcal{A}$  is non-nuclear, then  $d(\mathcal{A}) = 3$ .*

**PROOF.** Suppose  $\mathcal{A}$  is a separable unital  $C^*$ -algebra satisfying Condition (G): If  $\pi$  is a unital  $*$ -representation of  $\mathcal{A}$  on a Hilbert space  $K_1$  such that  $\pi(\mathcal{A})''$  is a type  $II_1$  factor, then  $\pi(\mathcal{A})''$  has Property  $\Gamma$ , where  $\pi(\mathcal{A})''$  is the von Neumann algebra generated by  $\pi(\mathcal{A})$  in  $B(K_1)$ .

Suppose  $\phi$  is a unital bounded homomorphism of  $\mathcal{A}$  on a Hilbert space  $K$ .

Since we just need to show that  $\|\phi\|_{cb} \leq \|\phi\|^3$ , we can focus on  $\phi(\mathcal{A})$  contained in the  $C^*$ -subalgebra of  $B(K)$  generated by  $\phi(\mathcal{A})$ , which is a separable  $C^*$ -algebra. Applying the GNS construction to a faithful state on the  $C^*$ -subalgebra generated by  $\phi(\mathcal{A})$  if necessary, we may just assume that  $K$  is a separable Hilbert space.

From Lemma 2.3,  $\phi$  can be extended to a bounded unital homomorphism  $\tilde{\phi} : \mathcal{A}^{\#} \rightarrow B(H)$  that is  $\sigma(\mathcal{A}^{\#}, \mathcal{A}^{\#}) \rightarrow \sigma(B(H), B(H)_{\#})$  continuous. Moreover  $\|\tilde{\phi}\| = \|\phi\|$  and  $\|\phi\|_{cb} \leq \|\tilde{\phi}\|_{cb}$ .

Let  $\mathcal{I} = \ker(\tilde{\phi})$ . Thus  $\mathcal{I}$  is a two-sided ideal of  $\mathcal{A}^{\#}$  and is closed in the  $\sigma(\mathcal{A}^{\#}, \mathcal{A}^{\#})$  topology. From Proposition 1.10.5 in [25], it follows that there is a projection  $p$  in the center of  $\mathcal{A}^{\#}$  such that  $\mathcal{I} = p\mathcal{A}^{\#}$ . Define  $\tilde{\phi}_p : (I - p)\mathcal{A}^{\#} \rightarrow B(K)$  as follows

$$\tilde{\phi}_p(a) = \tilde{\phi}(a), \quad \forall a \in (I - p)\mathcal{A}^{\#}.$$

Then  $\tilde{\phi}_p$  is a unital injective homomorphism, which is  $\sigma(\mathcal{A}^{\#}, \mathcal{A}^{\#})$  to  $\sigma(B(H), B(H)_{\#})$  continuous. Moreover,  $\|\tilde{\phi}\| = \|\tilde{\phi}_p\|$  and  $\|\tilde{\phi}\|_{cb} = \|\tilde{\phi}_p\|_{cb}$ .

We claim that  $(I - p)\mathcal{A}^{\#}$  is countably decomposable. Assume  $\{P_{\lambda}\}_{\lambda \in \Lambda}$  is a family of orthogonal projections in  $(I - p)\mathcal{A}^{\#}$  with sum  $I - p$ . Let  $\mathcal{B}$  be the von Neumann subalgebra generated by  $\{P_{\lambda}\}_{\lambda \in \Lambda}$  in  $(I - p)\mathcal{A}^{\#}$ . Obviously, Kadison's Similarity Problem for  $\mathcal{B}$  has an affirmative answer. In other words, there exists a positive invertible operator  $T$  in  $B(K)$  such that  $T\tilde{\phi}_p(\cdot)T^{-1}$  is a  $*$ -homomorphism from  $\mathcal{B}$  to  $B(K)$ , whence  $\{T\tilde{\phi}_p(P_{\lambda})T^{-1}\}_{\lambda \in \Lambda}$  is a family



of orthogonal projections in  $B(K)$ . From the facts that  $K$  is separable and  $\tilde{\phi}_p$  is injective, we conclude that  $\Lambda$  is a countable set and  $(I - p)\mathcal{A}^\#$  is countably decomposable.

Since  $\mathcal{A}$  embeds into  $\mathcal{A}^\#$  as a  $\sigma(\mathcal{A}^\#, \mathcal{A}^\#)$ -dense C\*-subalgebra, we know that  $(I - p)\mathcal{A}^\#$  is countably generated as a von Neumann algebra. Combining with the result that  $(I - p)\mathcal{A}^\#$  is countably decomposable, we obtain that  $(I - p)\mathcal{A}^\#$  has a separable predual. Without loss of generality, we might assume that  $(I - p)\mathcal{A}^\#$  acts on a separable Hilbert space  $H$ .

Suppose  $(I - p)\mathcal{A}^\#$  has a type decomposition

$$(I - p)\mathcal{A}^\# = \mathcal{M}_1 \oplus \mathcal{M}_{c_1} \oplus \mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty,$$

where  $\mathcal{M}_{c_1}$  is the type II<sub>1</sub> central summand. We can assume that  $\mathcal{M}_{c_1}$  acts on a separable Hilbert space  $H_1$ .

Notice that if  $\mathcal{M}_{c_1} = 0$ , then the proof of Theorem 5.2 will be simplified and

$$\|\phi\|_{cb} \leq \|\tilde{\phi}\|_{cb} \leq \|\tilde{\phi}\|^3 = \|\phi\|^3.$$

In the following we will assume that  $\mathcal{M}_{c_1}$  is nonvanishing and we show that  $\mathcal{M}_{c_1}$  is a type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ . Let  $\mathcal{M}_{\text{J}\infty} = \int_X \bigoplus \mathcal{M}_s d\mu$  and  $H_1 = \int_X \bigoplus H_s d\mu$  be the direct integral decompositions of  $\mathcal{M}_{c_1}$  and  $H_1$  relative to the center of  $\mathcal{M}_{c_1}$ . Thus  $\mathcal{M}_s$  is a type II<sub>1</sub> factor for almost  $s \in X$ . Note that  $\mathcal{A}$  embeds into  $\mathcal{A}^\#$  as a  $\sigma(\mathcal{A}^\#, \mathcal{A}^\#)$ -dense separable C\*-subalgebra. Also note  $p$  is a central projection of  $\mathcal{A}^\#$ . By Theorem 14.1.13 in [16], there is a unital \*-homomorphism  $\psi_s$  from  $(I - p)\mathcal{A}$  to  $\mathcal{M}_s$  such that  $\psi_s((I - p)\mathcal{A})$  generates  $\mathcal{M}_s$  as a von Neumann algebra for almost  $s \in X$ . Let  $\hat{\psi}_s : \mathcal{A} \rightarrow \mathcal{M}_s$  be such that  $\hat{\psi}_s(a) = \psi_s((I - p)a)$  for all  $a \in \mathcal{A}$ . Then  $\hat{\psi}_s$  is a \*-homomorphism from  $\mathcal{A}$  into  $\mathcal{M}_s$  such that  $\hat{\psi}_s(\mathcal{A})$  generates  $\mathcal{M}_s$  as a von Neumann algebra, for  $s \in X$  almost everywhere. Since  $\mathcal{A}$  satisfies condition (G),  $\mathcal{M}_s$  is a type II<sub>1</sub> factor with Property  $\Gamma$ , for  $s \in X$  almost everywhere. From Proposition 3.12 in [23], it induces that  $\mathcal{M}_{c_1}$  is a type II<sub>1</sub> von Neumann algebra with Property  $\Gamma$ .

Since  $\tilde{\phi}_p : (I - p)\mathcal{A}^\# \rightarrow B(K)$  is a unital homomorphism, which is  $\sigma(\mathcal{A}^\#, \mathcal{A}^\#)$  to  $\sigma(B(H), B(H)_\#)$  continuous, by Theorem 5.2, we have that  $\|\tilde{\phi}_p\|_{cb} \leq \|\tilde{\phi}_p\|^3$ , whence

$$\|\phi\|_{cb} \leq \|\tilde{\phi}\|_{cb} = \|\tilde{\phi}_p\|_{cb} \leq \|\tilde{\phi}_p\|^3 = \|\tilde{\phi}\|^3 = \|\phi\|^3$$

and  $d(\mathcal{A}) \leq 3$ .

It was shown in [21] that the similarity degree of a C\*-algebra is less than or equal to 2 if and only if the C\*-algebra is nuclear. Notice that the similarity degree is always an integer. Since  $d(\mathcal{A}) \leq 3$ , we know if  $\mathcal{A}$  is non-nuclear, then  $d(\mathcal{A}) = 3$ . The proof is complete.  $\square$

**DEFINITION 5.4.** *A unital C\*-algebra  $\mathcal{A}$  is said to have Property  $c^*\text{-}\Gamma$  if it satisfies Condition (G) in Theorem 5.3, as follows: If  $\pi$  is a unital \*-representation of  $\mathcal{A}$  on a Hilbert space  $H$  such that  $\pi(\mathcal{A})''$  is a type II<sub>1</sub> factor, then  $\pi(\mathcal{A})''$  has Property  $\Gamma$ , where  $\pi(\mathcal{A})''$  is the von Neumann algebra generated by  $\pi(\mathcal{A})$  in  $B(H)$ .*

**COROLLARY 5.5.** *Let  $\mathcal{A}$  be a separable unital C\*-algebra. Suppose  $\mathcal{B}$  is a nuclear separable unital C\*-algebra with no finite dimensional representations. Then  $\mathcal{A} \otimes_{\min} \mathcal{B}$  has Property  $c^*\text{-}\Gamma$  and  $d(\mathcal{A} \otimes_{\min} \mathcal{B}) \leq 3$ , where  $\mathcal{A} \otimes_{\min} \mathcal{B}$  is the minimal tensor product of  $\mathcal{A}$  and  $\mathcal{B}$ .*

PROOF. By Theorem 5.3, we only need to prove the following statement: *if  $\pi$  is a representation of  $\mathcal{A} \otimes_{\min} \mathcal{B}$  on a Hilbert space  $H$  and  $\pi(\mathcal{A} \otimes_{\min} \mathcal{B})''$  is a type  $II_1$  factor, then  $\pi(\mathcal{A} \otimes_{\min} \mathcal{B})''$  has Property  $\Gamma$ .*

Assume that  $\pi$  is a representation of  $\mathcal{A} \otimes_{\min} \mathcal{B}$  on a Hilbert space  $H$  and  $\mathcal{M} = (\pi(\mathcal{A} \otimes_{\min} \mathcal{B}))''$  is a type  $II_1$  factor.

Since  $I_{\mathcal{A}} \otimes \mathcal{B}$  is a nuclear  $C^*$ -algebra with no finite dimensional representation, we get that  $\pi(I_{\mathcal{A}} \otimes \mathcal{B})''$  is an infinite dimensional hyperfinite von Neumann algebra. By the fact that  $\pi(I_{\mathcal{A}} \otimes \mathcal{B})$  commutes with  $\pi(\mathcal{A} \otimes I_{\mathcal{B}})$ , it follows that  $\pi(I_{\mathcal{A}} \otimes \mathcal{B})''$  is a hyperfinite type  $II_1$  factor.

Note that (i)  $\mathcal{M}$  is the type  $II_1$  factor generated by commuting  $C^*$ -subalgebras  $\pi(\mathcal{A} \otimes I_{\mathcal{B}})$  and  $\pi(I_{\mathcal{A}} \otimes \mathcal{B})''$ , and (ii)  $\pi(I_{\mathcal{A}} \otimes \mathcal{B})''$  is a hyperfinite type  $II_1$  factor. A standard argument shows that  $\mathcal{M}$  has Property  $\Gamma$  and the proof of the corollary is complete.  $\square$

Since the Jiang-Su algebra  $\mathcal{Z}$  (see definition in [12]) is nuclear, simple and infinite dimensional, we obtain the next corollary directly.

COROLLARY 5.6. *If a separable, non-nuclear, unital  $C^*$ -algebra  $\mathcal{A}$  is  $\mathcal{Z}$ -stable, then  $d(\mathcal{A}) = 3$ .*

EXAMPLE 5.7. *Let  $F_2$  be a non-abelian free group on two generators  $a, b$  and  $C_r^*(F_2)$  be the reduced  $C^*$ -algebra of free group factor on two standard generators  $\lambda(a), \lambda(b)$ . Let  $\text{Aut}(C_r^*(F_2))$  be the automorphism group of  $C_r^*(F_2)$ . Let  $\theta$  be an irrational number. Let  $\mathbb{Z}$  be the group of integers and  $g$  be a generator of  $\mathbb{Z}$ . Let*

$$\alpha : \mathbb{Z} \rightarrow \text{Aut}(C_r^*(F_2))$$

*be a group homomorphism from  $\mathbb{Z}$  to  $\text{Aut}(C_r^*(F_2))$  defined by the following action:*

$$\alpha(g)(\lambda(a)) = e^{2\pi i \cdot \theta} \lambda(a) \quad \text{and} \quad \alpha(g)(\lambda(b)) = e^{2\pi i \cdot \theta} \lambda(b).$$

*Let  $C_r^*(F_2) \rtimes_{\alpha} \mathbb{Z}$  be the reduced crossed product of  $C_r^*(F_2)$  by an action  $\alpha$  of  $\mathbb{Z}$ . Then  $C_r^*(F_2) \rtimes_{\alpha} \mathbb{Z}$  has Property  $c^*$ - $\Gamma$  (see [11]) and  $d(C_r^*(F_2) \rtimes_{\alpha} \mathbb{Z}) = 3$ .*

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